

ANALOG AND DIGITAL SIGNAL PROCESSING

CHAPTER 3 : LINEAR SYSTEM RESPONSE (GENERAL CASE)

3. Linear System Response (general case)

3.1 INTRODUCTION

In chapter 2, we determined that :

- If the system is linear (or operate in a linear domain)
- If the input signal can be assumed as periodic over the interval of time considered
- If the transients have died down from a sufficient period of time

➔ The complex numbers and the superposition principle can be used in order to compute the output signal of the considered system.

In this chapter, we consider the linear system response in the general case (I.e. transient and non-periodic).

Example 3.1

The input signal $u_1(t)$ of the circuit of Fig. 2.8 (R-L-C band-pass filter) is given by the following relations and plotted on Fig 3.1 :

$$u_1(t) = \begin{cases} 0 & t < 5 \text{ ms} \\ \cos(\omega_x t) e^{-(t-0.005)/0.005} & 5 \text{ ms} < t < 15 \text{ ms} \\ 2 \cos(\omega_x t) e^{-(t-0.015)/0.01} & 15 \text{ ms} < t < 25 \text{ ms} \\ 3 \cos(\omega_x t) & 25 \text{ ms} < t \end{cases} \quad (f_x = 500 \text{ Hz})$$

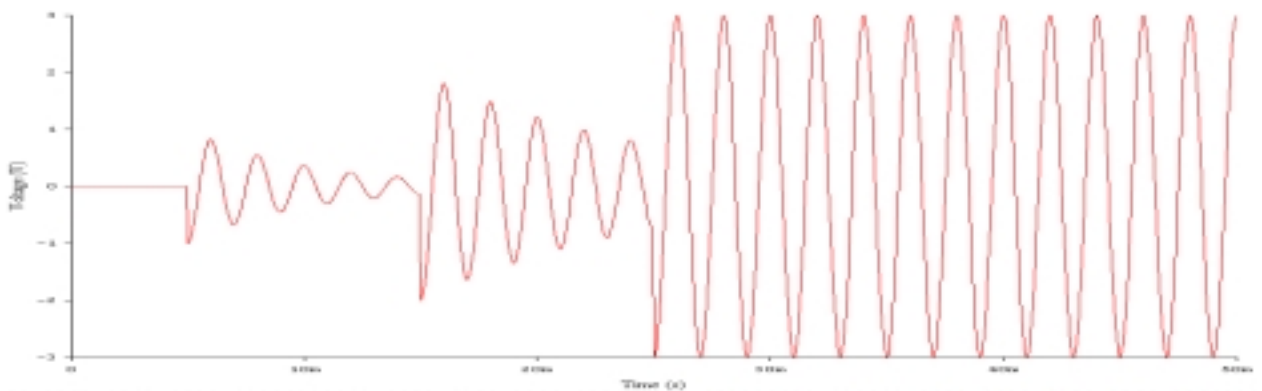


Figure 3.1 $u_1(t)$

How to determine $u_2(t)$?

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a) Computer simulation (EWB)

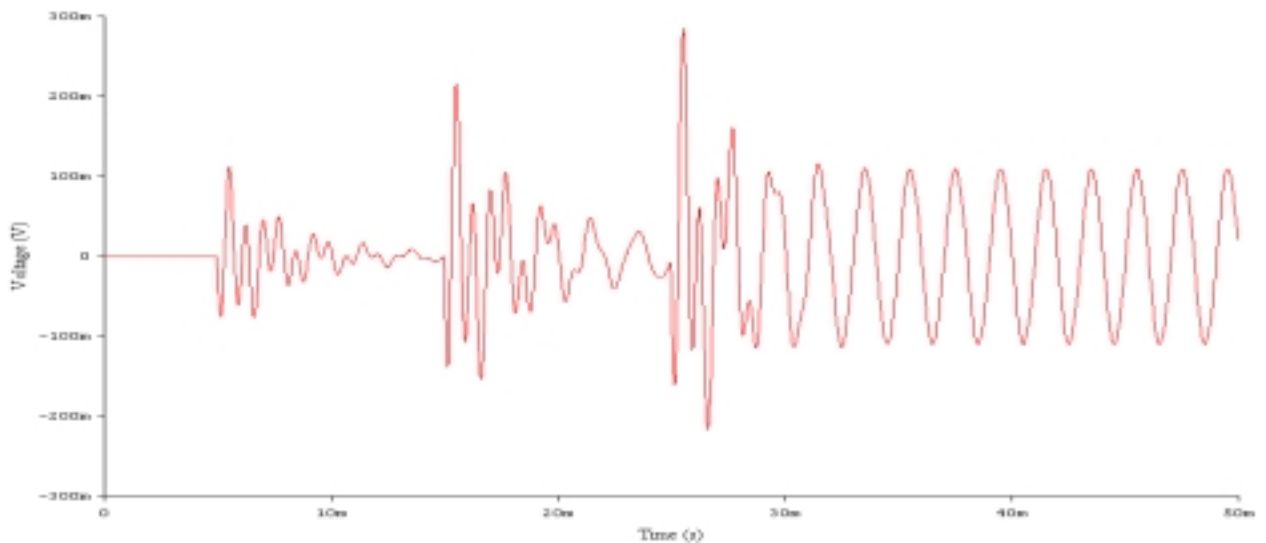


Figure 3.2 $U_2(t)$, EWB

Note : transient response : for $0 < t < 30$ ms
steady state response : for $t > 30$ ms

b) Convolution

Convolve the input signal $u_1(t)$ with the impulse response of the filter $h(t)$.

$$u_2(t) = u_1(t) * h(t)$$

c) Laplace transform

$$u_2(t) = L^{-1}[L[u_1(t)] \cdot L[h(t)]]$$

d) Differential Equations

3.2 TRANSIENT RESPONSE : GENERAL PRINCIPAL

In order to test the behavior of a newly designed system, there is a class of elementary test signals frequently used in system analysis. Its members have a very simple mathematical form but are discontinuous or have discontinuous derivatives. They are usually referred as *singularity functions*.

Three of the most common singularity functions are the impulse function, the unit step function and the ramp function as shown in the next figure.

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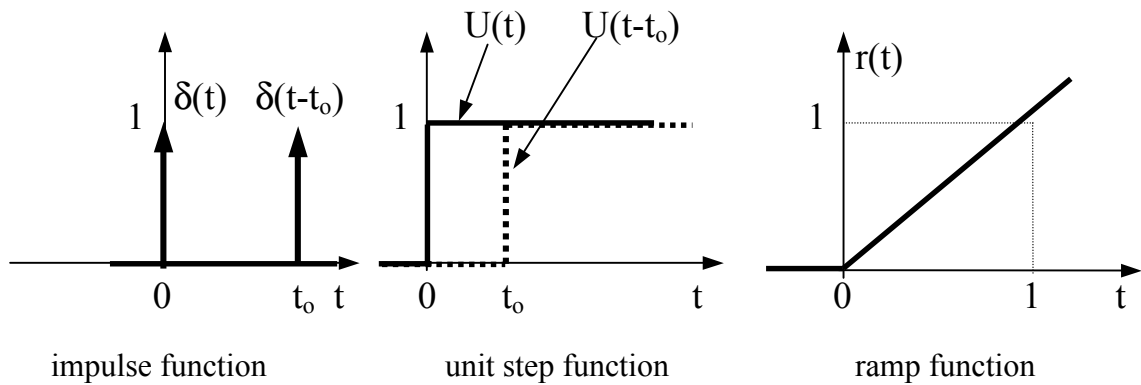


Figure 3.3 Singulariry functions

3.2.1 IMPULSE FUNCTION (DIRAC DELTA FUNCTION)

The intuitive interpretation of the impulse function is that it is an idealization of a very narrow pulse having a finite total area.

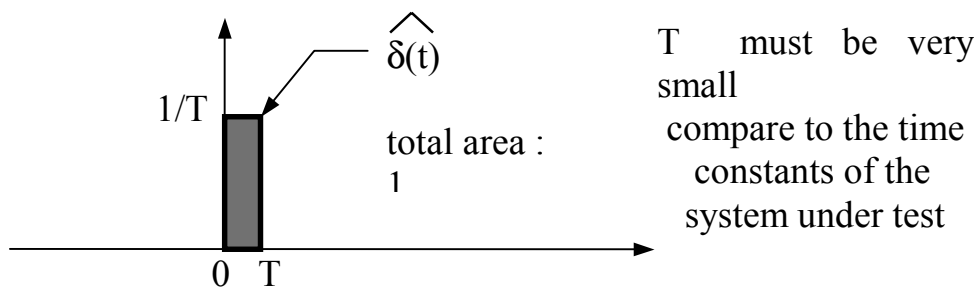


Figure 3.4 Approximation of the impulse function

The idealized impulse function has some of the following properties :

$$\delta(t-t_0) = 0 \quad \text{for } t \neq t_0 \quad (T \rightarrow 0)$$

$$\int_{t_1}^{t_2} \delta(t - t_0) dt = 1 \quad \text{if } t_1 < t_0 < t_2$$

$$\int_{-\infty}^{\infty} f(t) \cdot \delta(t - t_0) dt = f(t_0) \quad \text{with } f(t) \text{ continuous at } t_0$$

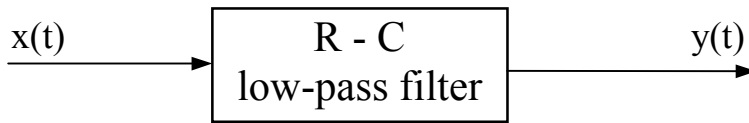
$$\delta(t) = \lim_{a \rightarrow 0} \frac{u(t) - u(t - a)}{a}$$

$$\delta(t) = \frac{d}{dt} u(t) \quad \text{Note that the derivative of the step function does not exist in the strict mathematical sense}$$

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Application example of the impulse function



The frequency response is :
$$H(j\omega) = \frac{1}{1 + j\omega\tau}$$

Hence the Laplace transform is :
$$H(s) = \frac{1}{1 + s\tau} = \frac{\frac{1}{\tau}}{s + \frac{1}{\tau}}$$

and the inverse Laplace Transform gives :
$$h(t) = \frac{1}{\tau} \cdot e^{-\frac{t}{\tau}} \cdot u(t) \quad \text{Impulse Response}$$

If the excitation $x(t)$ approximates an impulse function, then $y(t)$ is equal to $h(t)$ since the inverse Laplace transform is the impulse response of the system. A rule of thumb is to say that the duration T of the pulse approximating an impulse has to be at least 10 times smaller than the smallest time constant involved in the system under test. In the above example (R-C low-pass filter), let admit that τ is equal to 1 ms. Then T can be taken in the order of 0.1 ms. Thus, its amplitude must be scaled up by a factor of 10000, thus transforming a normalized 1 V impulse into a 10 KV pulse of 0.1 ms duration !! By choosing a more realistic pulse amplitude of 10 V, the measured approximated system impulse response will be scaled down by a factor of 1000.

However, in our example, from $h(t)$ we determine that its maximum value occurring at $t=0$, is $1/\tau$ or 1000. Thus with our approximated impulse, $y(0) = 1$ V, an easily measurable value with any standard oscilloscope !

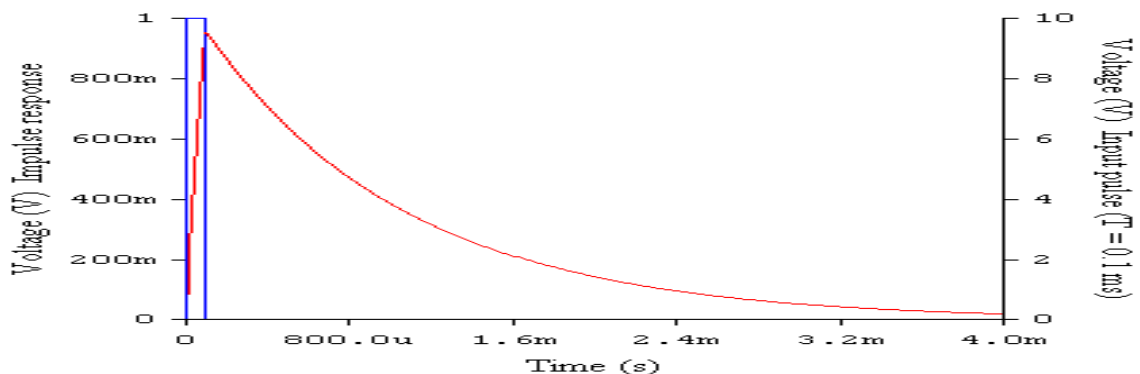


Figure 3.5 Impulse response approximation

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Problem (Impulse response)

Consider the following circuit and its approximate impulse response :

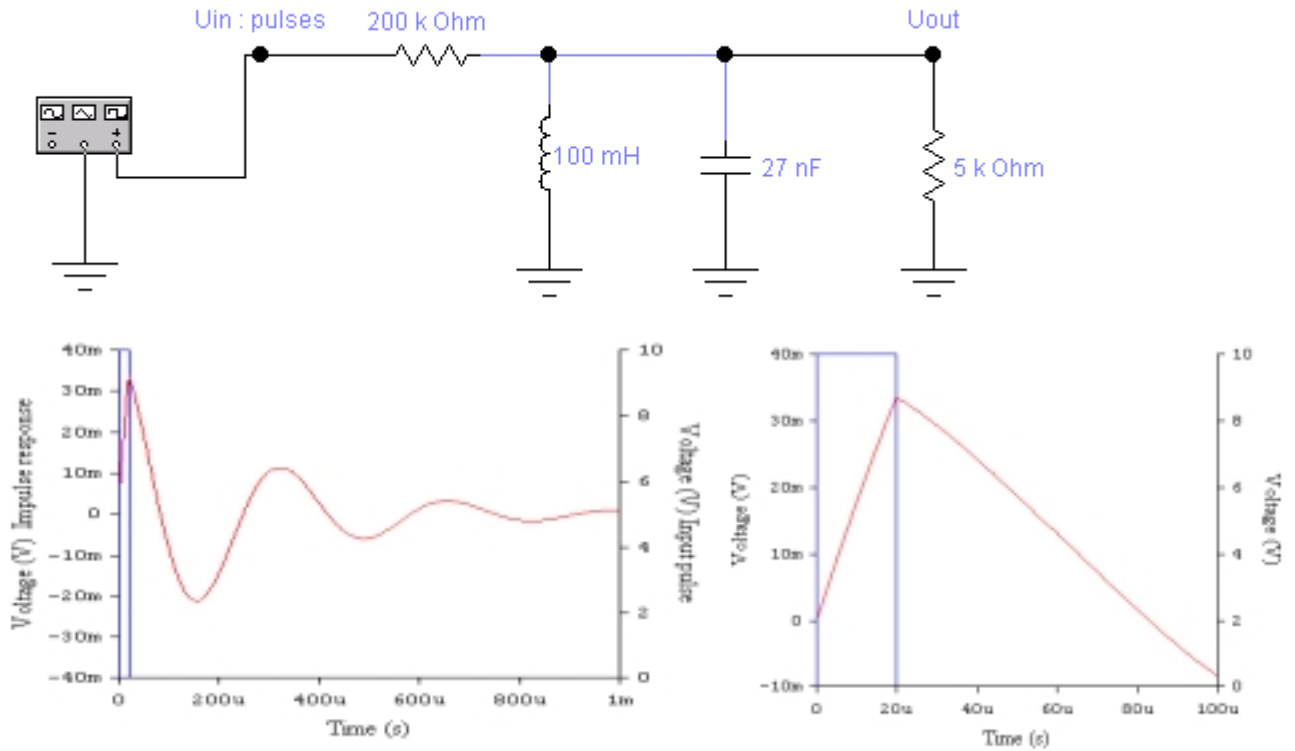


Figure 3.6 Band-pass filter under test and its Impulse response approximation

What kind of problem do you encounter and how do you practically solve it if you desire to accurately determine an impulse response like that?

Solution :

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3.2.2 UNIT STEP AND RAMP FUNCTION

The unit step function designated as $u(t)$, is defined to be zero before zero time and unity thereafter. Thus, it may be represented mathematically as :

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

The ramp function is just the integral of the step function and conversely, the derivative of $r(t)$ is $u(t)$ excepted at $t=0$ where a unique derivative does not exist. By using combinations of time shifted ramp and step functions, many other type of functions can be represented.

Example

A time shifted rectangular pulse can be written as :

$$p_{a,b}(t) = u(t-a) - u(t-b) \quad \text{with } b > a > 0$$

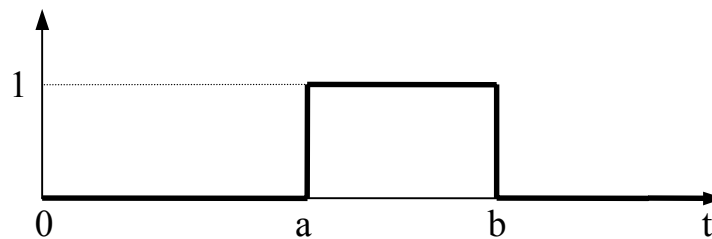


Figure 3.7

Application example of the unit step

Consider the following circuit :

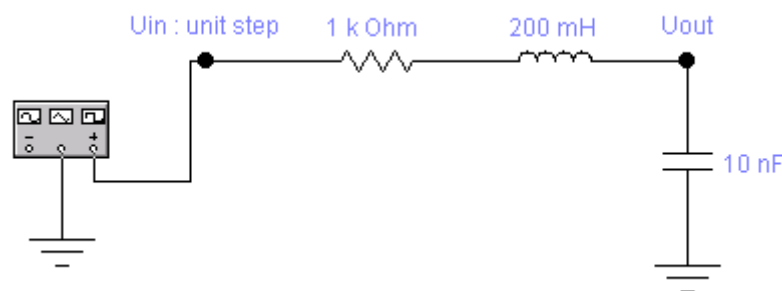


Figure 3.8

- What kind of filter is it ?
- Determine its unit step response.

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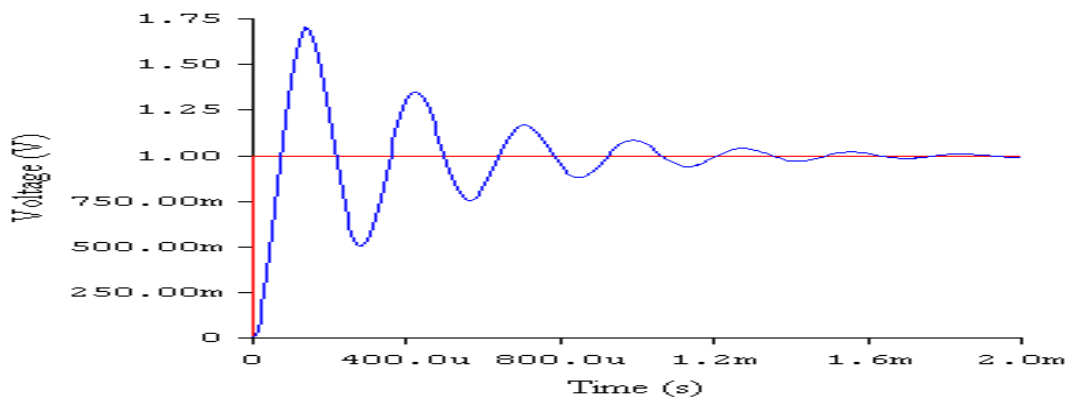


Figure 3.9 Unit step response (ringing effect !) with EWB

This response can also be computed with the Laplace transform or the convolution (see next sections).

3.2.3 OTHER FREQUENTLY USED TEST SIGNALS

There are a few other test signals that are commonly used. One of them is called « Chirp », it consists in a sine-wave whose frequency is swept at a given speed between two predefined frequencies (f_{low} and f_{high}).

Example (Chirp) If we apply a chirp (Fig. 3.10) to the band-pass filter depicted in Fig. 2.8 , than we obtain the output signal $u_2(t)$ given in Fig. 3.11.

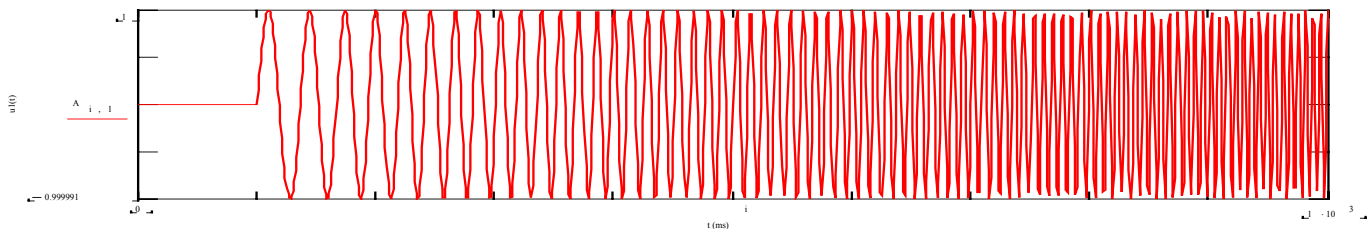


Figure 3.10 $u_1(t)$ Chirp

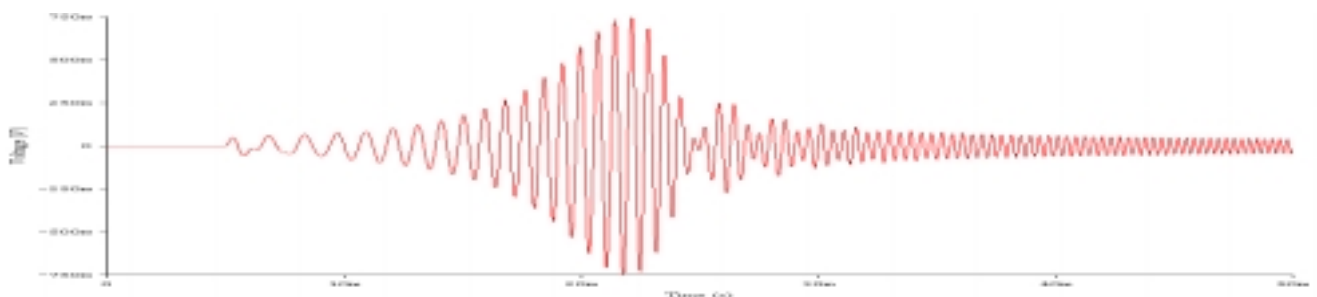


Figure 3.11 $u_2(t)$

3.2.4 FILTER RISE AND FALL TIME

In fig. 3.12, rise time of three different bandwidths (60, 200 and 600 Hz) of a Butterworth band-pass filters (center freq.: 1000 Hz) are compared.

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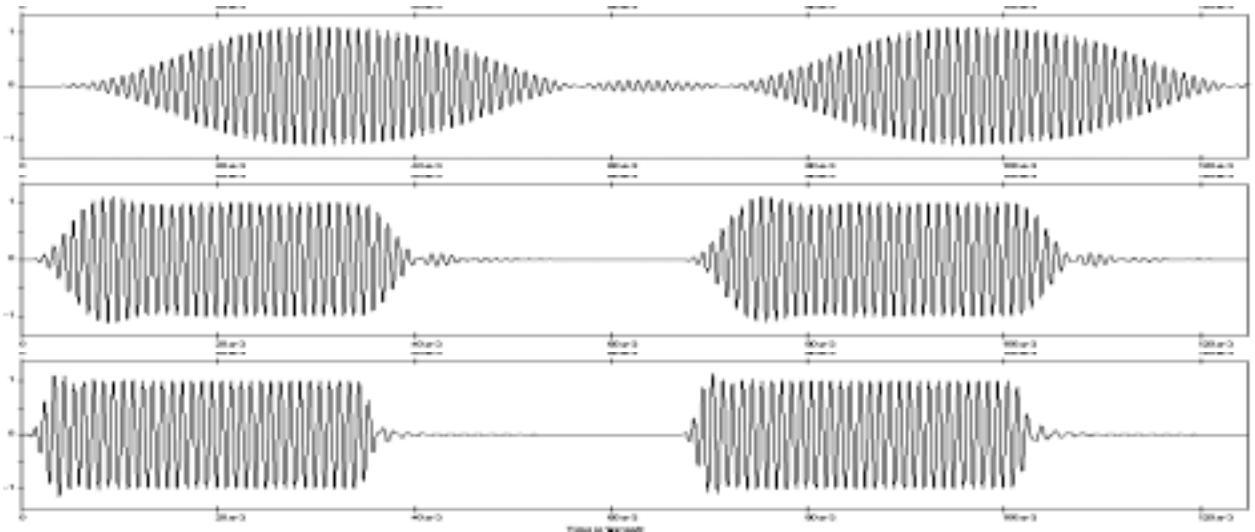


Figure 3.12 Top to bottom: Butterworth BPF output (Bw: 60, 200 and 600 Hz)

From this figure, the 60 Hz BPF bandwidth estimated rise time is of 15 ms, of 5 ms for 200 Hz and of 1.5 ms for 600 Hz. As a rule of thumb, we can tell that :

$$t_{\text{rise}}(\text{in second}) = t_{\text{fall}} = K_{Bw} / Bw(\text{in Hz}) , 0.7 < K < 1.3 \quad (K_{Bw}=1)$$

The same experiment applied to low-pass filters of cut-off frequencies $f_{\text{cut-off}}$ of 60, 200 and 600 Hz give rise times of 10, 3 and 1 ms respectively. Thus :

$$t_{\text{rise}}(\text{in second}) = t_{\text{fall}} = K_{co} / f_{\text{cut-off}}(\text{in Hz}) , 0.4 < K_{co} < 0.8 \quad (K_{co}=0.6)$$

Fig. 3.13 shows the impact of the filter bandwidth on the noise reduction. With **Wide-Band** noise, the **Noise Power (Pn)** is proportional to the bandwidth

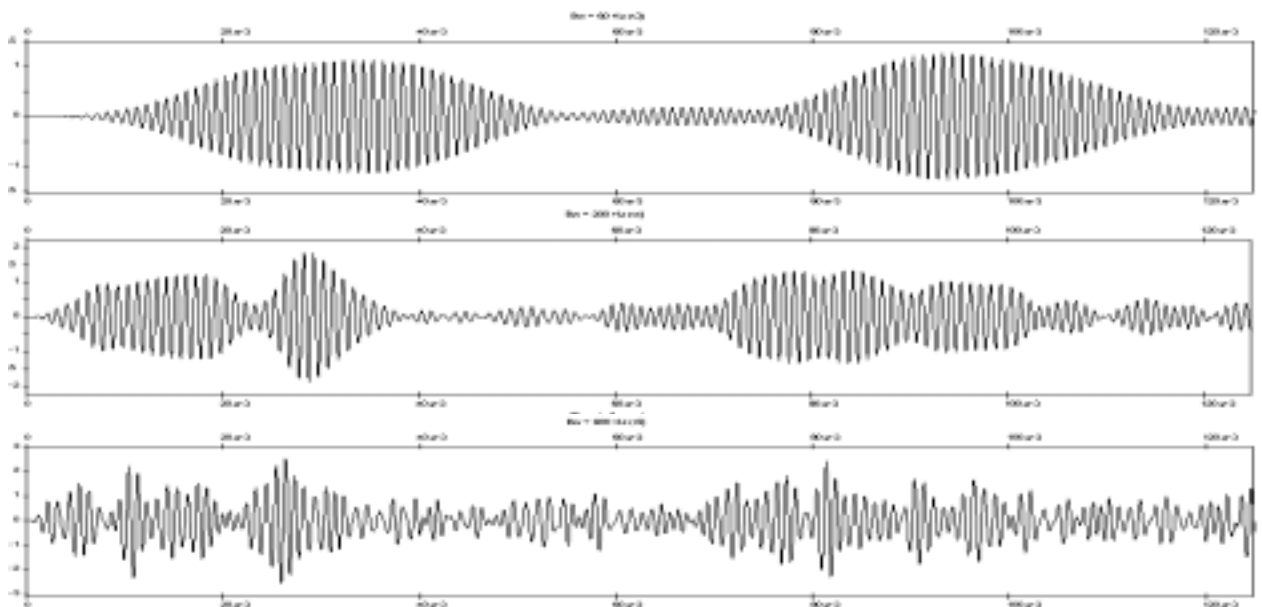


Figure 3.13 Filter Input: ON-OFF Mod. + Additive Wide-Band Gaussian Noise

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3.3 CONVOLUTION

The convolution is a technique used to solve the following problem :

If $x(t)$, the excitation of a linear system whose impulse response $h(t)$ is known, how do we mathematically determine its output $y(t)$?

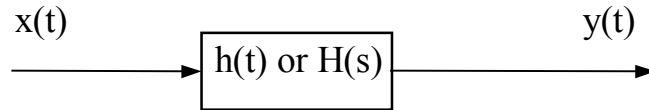


Figure 3.14

We know that if $x(t)$ is an impulse, then $y(t)$ is $h(t)$, that is:

$$x(t) = \delta(t-T) \implies y(t) = h(t-T)$$

$x(t)$ can be approximated by a train of impulses : $x(t) = \sum_{n=-\infty}^{\infty} A_n \cdot \delta(t - nT)$

then $y(t)$ becomes : $y(t) = \sum_{n=-\infty}^{\infty} A_n \cdot h(t - nT)$

Example : Triangular input signal $x(t)$ represented by a train of 7 impulses.

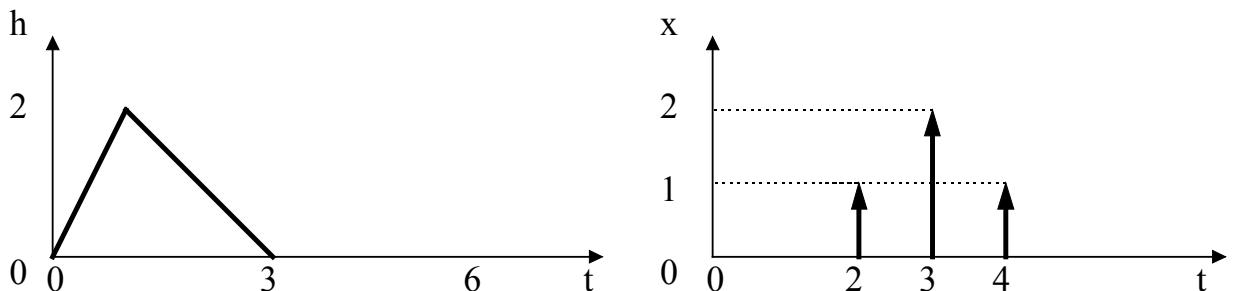
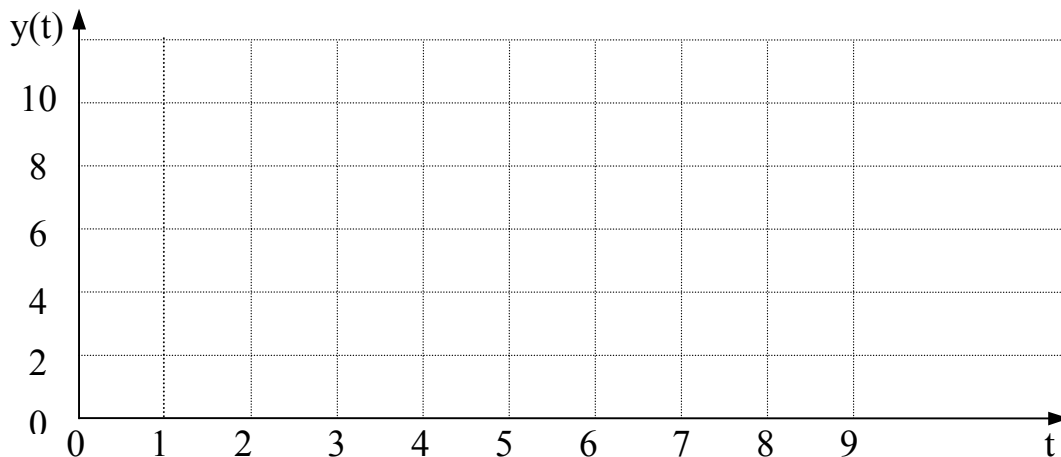


Figure 3.15



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3.3.1 MATHEMATICAL REPRESENTATION OF THE CONVOLUTION

If $x(t)$ is continuous, then $y(t)$ can be mathematically computed by what is called the **CONVOLUTION**.

$$y(t) = x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda) \cdot x(t - \lambda) d\lambda = \int_{-\infty}^{\infty} x(\lambda) \cdot h(t - \lambda) d\lambda$$

3.3.2 APPLICATION EXAMPLE OF THE CONVOLUTION

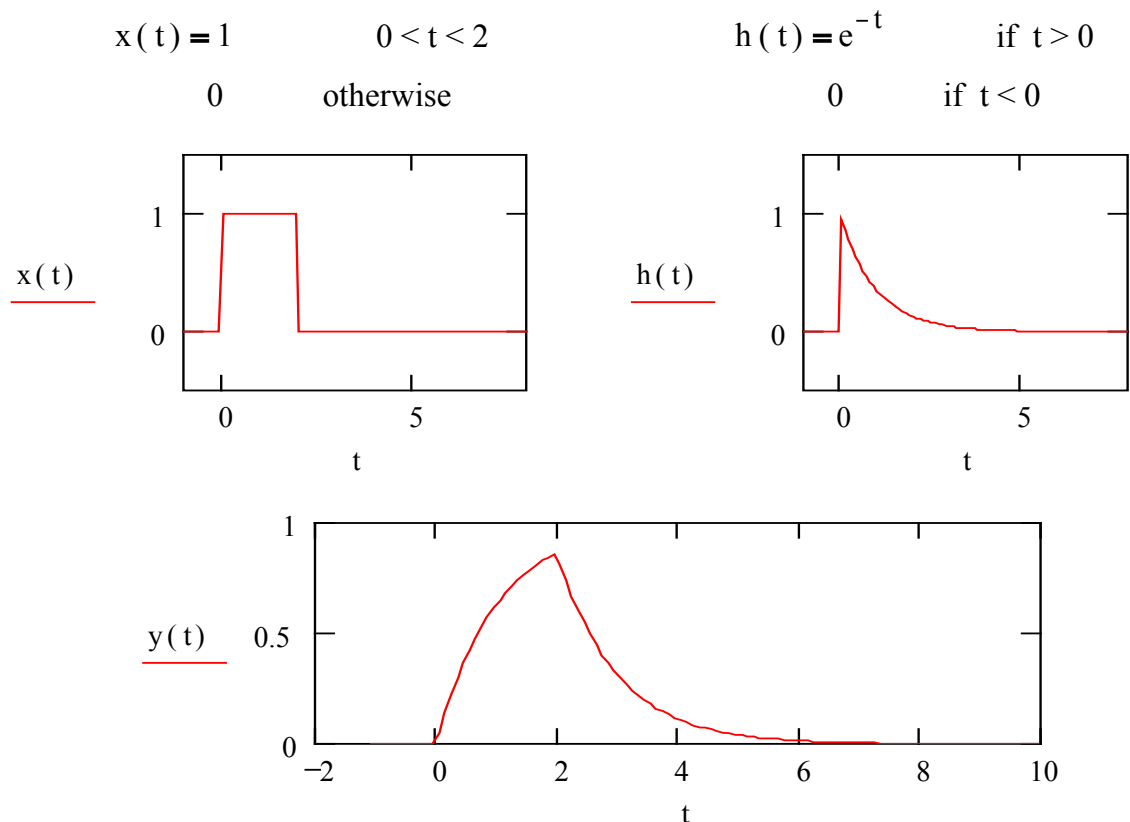


Figure 3.16

How to mathematically represent $y(t)$?

Solution :

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3.3.2 A APPLICATION EXAMPLES OF THE CONVOLUTION

Example A

$$x(t) = \delta(t-2) + \delta(t+2)$$

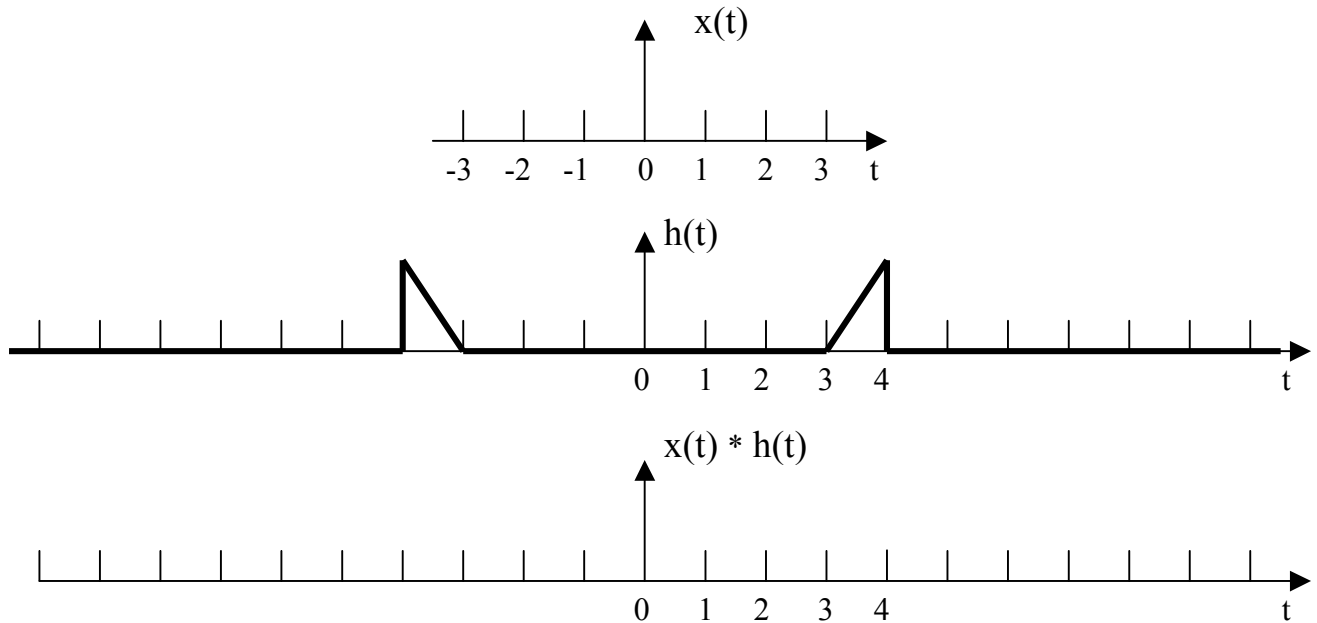


Figure 3.17

Example B

$$x(t) = \delta(t-2) + \delta(t+2)$$

$$h(t) = \delta(t-6) + \delta(t+6)$$

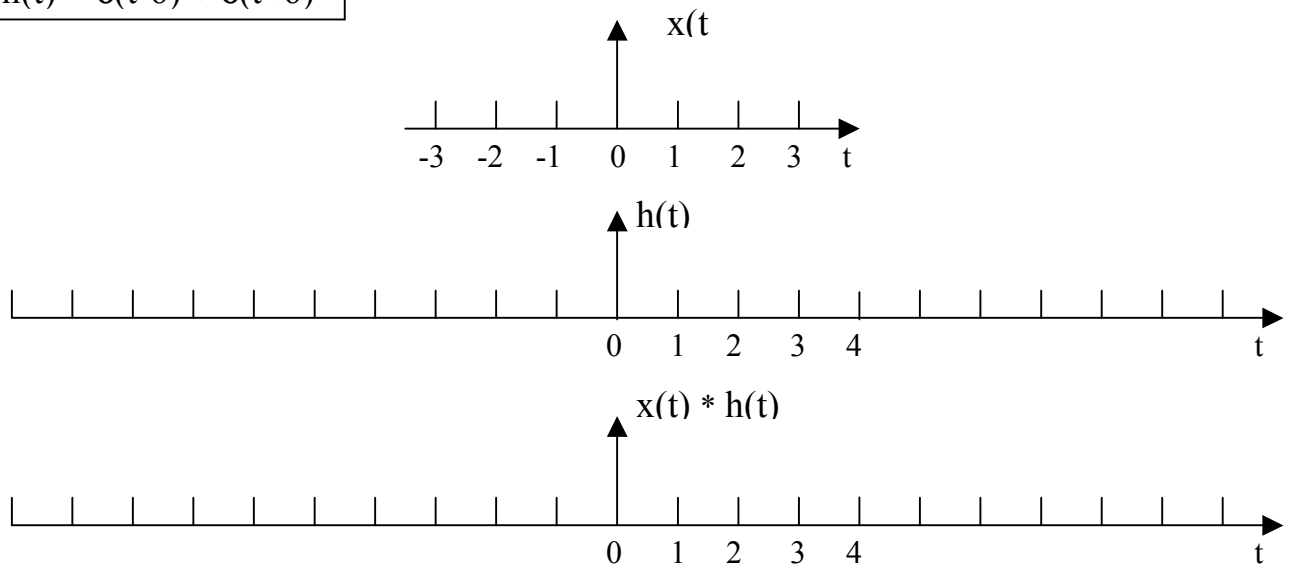


Figure 3.18

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APPLICATION EXAMPLES OF THE CONVOLUTION (CON'T)

Example C

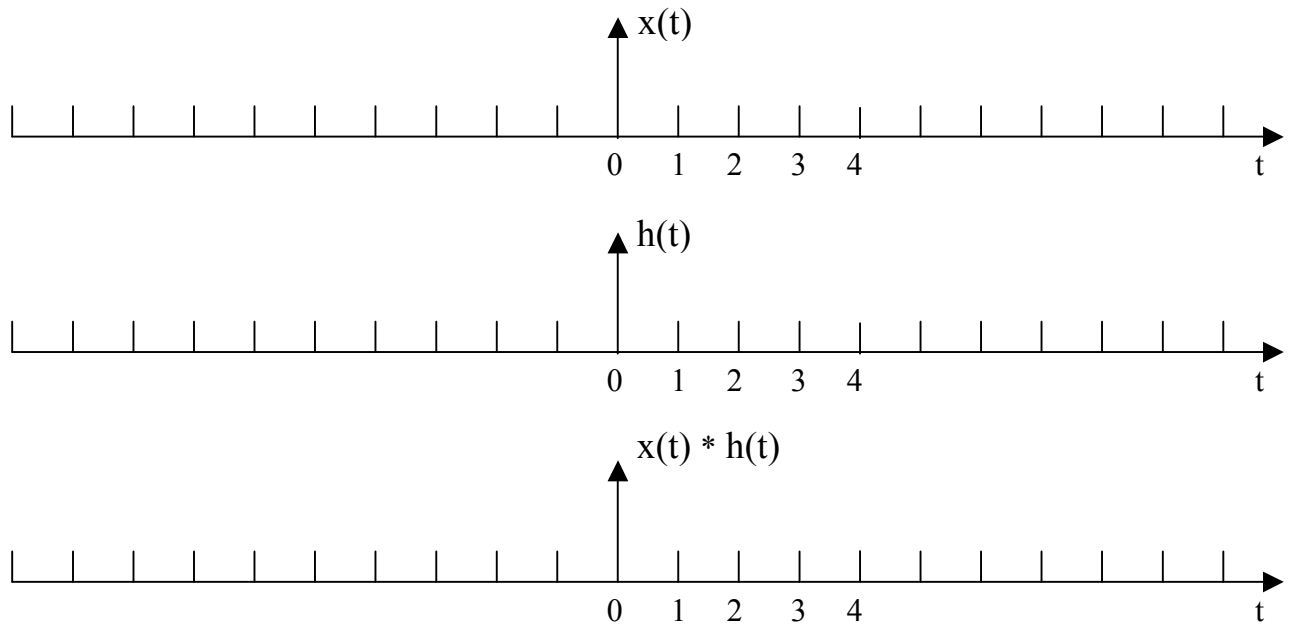


Figure 3.19

Example D

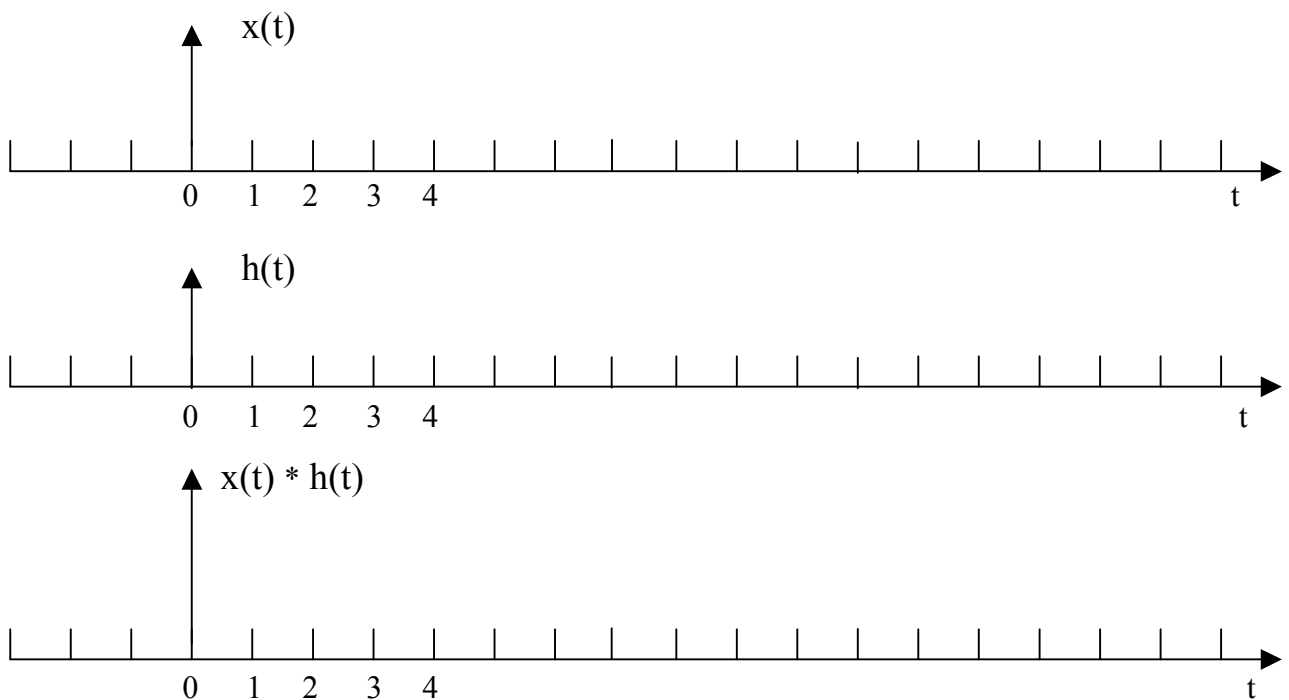


Figure 3.20

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APPLICATION EXAMPLES OF THE CONVOLUTION (CON'T)

Example E : Running Averaging

$h(t)$ is a "Running Averager", i.e. $h(t) = U(t) - U(t-T)$

$h_1(t) : T = 0.5 \text{ s}$

$h_2(t) : T = 2.0 \text{ s}$

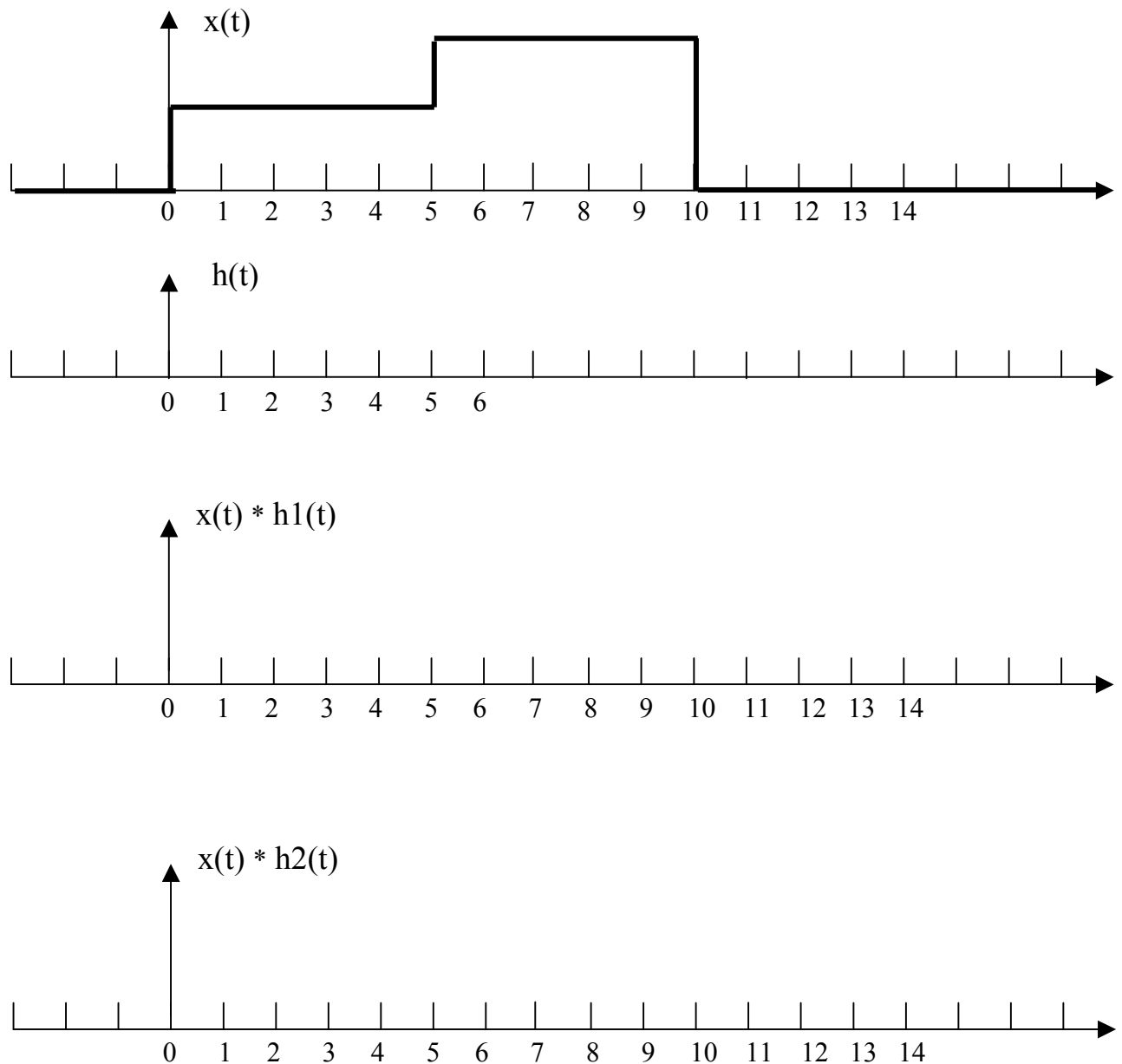


Figure 3.21

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3.3.3 PROPERTIES OF THE CONVOLUTION

When the functions to convolve are reasonably well behaved, the following properties hold :

Distributivity - Associativity – Commutativity

An additional interesting property is the following :

$$\text{if } y(t) = x(t) * h(t) \quad \text{then} \quad y'(t) = x'(t) * h(t) = x(t) * h'(t)$$

Example (Deconvolution)

A signal is recorded on a film by varying the intensity of light passing through a narrow slit and falling on a moving film. From the recorded light intensity $y(t)$, find $x(t)$, or in other word, deconvolve $y(t)$ knowing that the slit is rectangular and that the film velocity v is constant and of known value. This problem is schematically shown in the next figure :

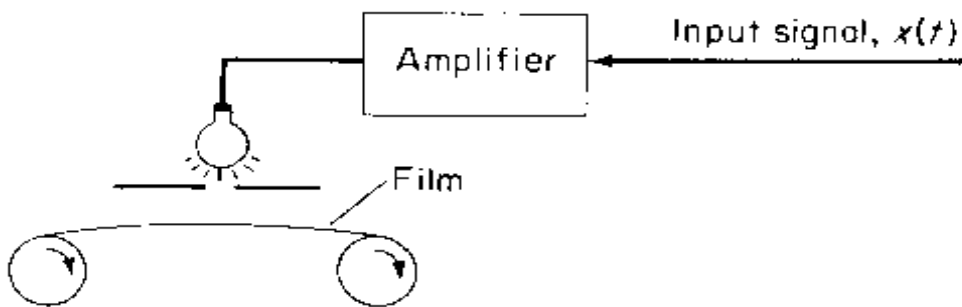


Figure 3.22

Making use of the solution of the convolution of a function with a pulse representing the window effect :

$$y(t) = x(t) * [u(t+T/2) - u(t-T/2)] \quad (\text{running average})$$

Using the derivative relationship given previously, we obtain :

$$\begin{aligned} y'(t) &= x(t) * [u'(t+T/2) - u'(t-T/2)] \\ &= x(t) * [\delta(t+T/2) - \delta(t-T/2)] = x(t+T/2) - x(t-T/2) \end{aligned}$$

Therefore, solving for $x(t)$, we have :

$$x(t+T/2) = y'(t) + x(t-T/2) \quad \rightarrow \quad \mathbf{x(t) = y'(t-T/2) + x(t-T)}$$

The value of $x(t)$ is given by the derivative of $y(t)$ one-half a slit width earlier, plus the value of $x(t)$ obtained one full slit width earlier.

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3.3.4 NOTES : FOLDING AND SLIDING OF A FUNCTION

The next figures illustrate graphically how to time-shift (slide) and fold a function; an important concept particularly useful in convolution.

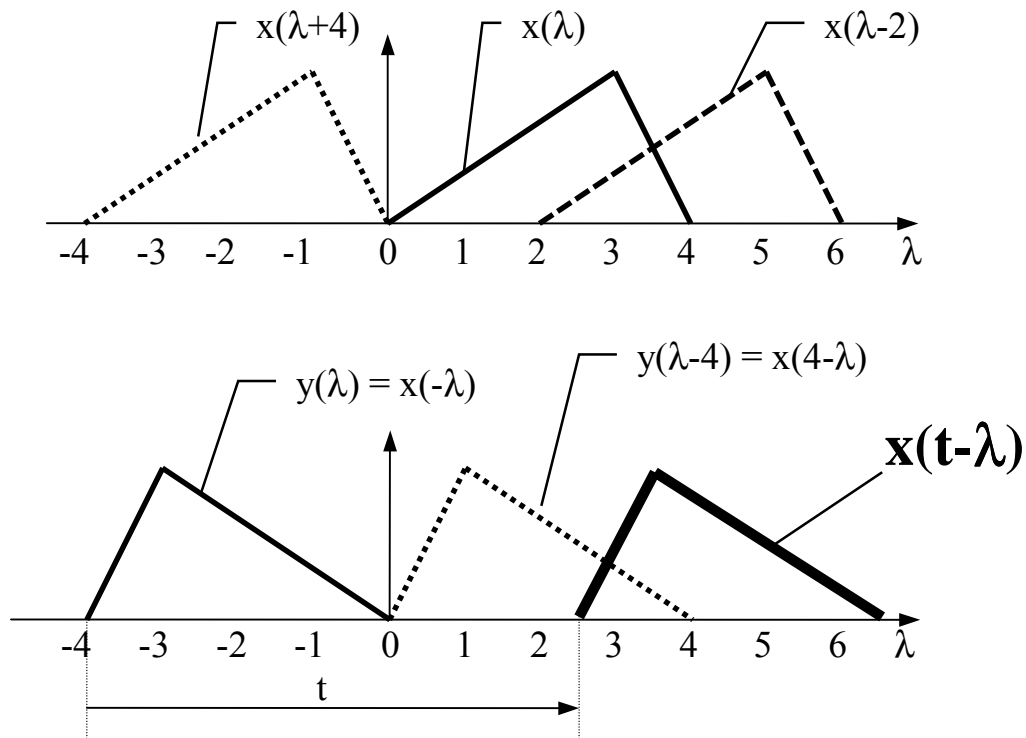


Figure 3.23

3.3.5 OTHER APPLICATIONS OF THE CONVOLUTION

One property of the Fourier transform tells that if two signals $x_1(t)$ and $x_2(t)$ are multiplied, the Fourier transform of the output $y(t)$ is obtained in convolving the Fourier transforms of the two inputs.

$$y(t) = x_1(t) \cdot x_2(t) \quad \rightarrow \quad F[y(t)] = F[x_1(t)] * F[x_2(t)]$$

This result is very important in many Communication Applications (i.e. On-Off modulated sine-wave, FSK).

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3.4 CONTINUOUS FOURIER TRANSFORM

The Fourier series expansion is usually a good model for periodic waveforms. However, in real application most signals are periodic over a limited time period or not periodic at all. In order to apprehend this problem, a new form of representation has to be used.

The basic idea is described in the following figure :

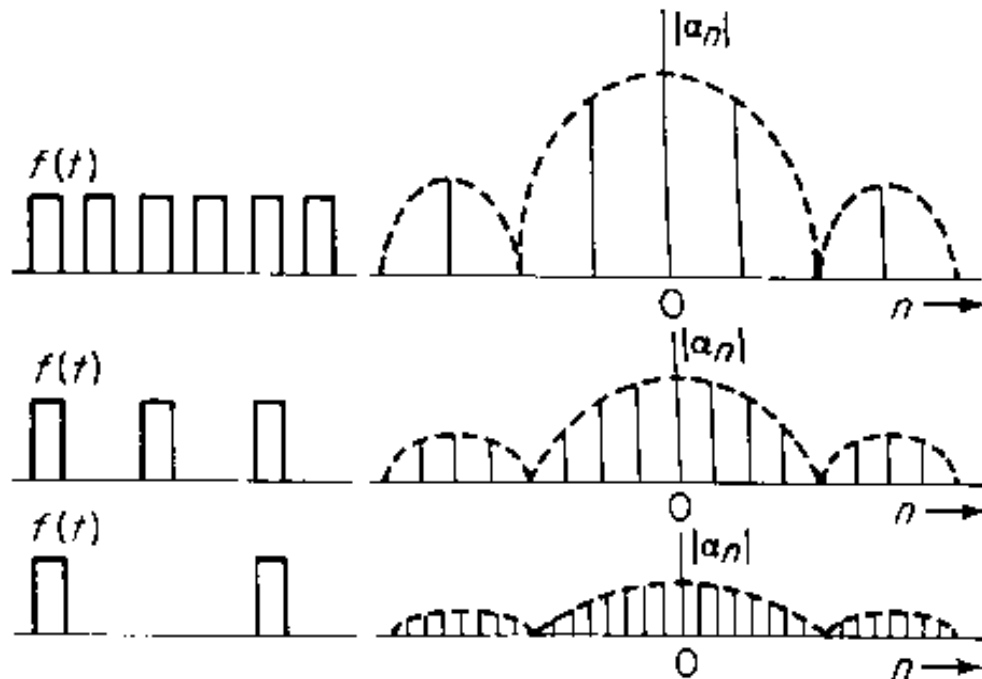


Figure 3.24 Spectrum of pulse train with increasing period

This figure shows the effect on the line spectrum $|a_n|$ as the period is increased from T to $2T$ to $4T$. There are three important observations that can be made :

- 1) The relative shape of the spectrum is not affected by the change of the period but is entirely determined by the shape of the pulse.
- 2) The only way the period T influences the spectrum is by altering the amplitude in a manner inversely proportional to the period T .
- 3) The spectral lines become denser as T is increased since these lines are separated by a 'distance' that is inversely proportional to the period T .

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Thus, when $T \rightarrow \infty$, the summation of the Fourier series passes into an *integral*. This *Fourier Integral* is called the *Fourier Transform*. The formal mathematical definition is the following :

Fourier Transform	$F(f(t)) = F(\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j \cdot \omega \cdot t} dt$
Inverse Fourier Transform	$f(t) = \text{Inv}F(\omega) = \frac{1}{2 \cdot \pi} \cdot \int_{-\infty}^{\infty} F(\omega) \cdot e^{j \cdot \omega \cdot t} d\omega$

Example

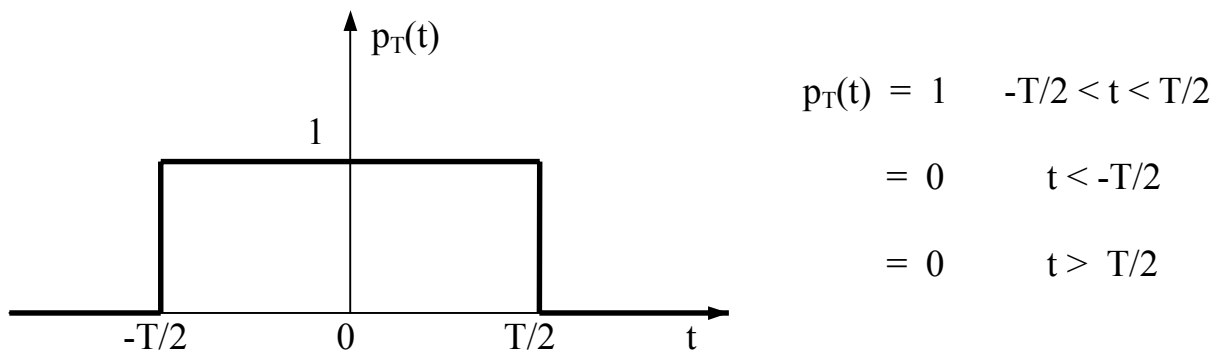


Figure 3.25 Rectangular pulse

$$P_T(\omega) = \int_{-\infty}^{\infty} p_T(t) \cdot e^{-j \cdot \omega \cdot t} dt = \int_{-T/2}^{T/2} e^{-j \cdot \omega \cdot t} dt$$

$$P_T(\omega) = \frac{e^{j \cdot \omega \cdot \frac{T}{2}} - e^{-j \cdot \omega \cdot \frac{T}{2}}}{j \cdot \omega} = T \cdot \frac{\sin\left(\omega \cdot \frac{T}{2}\right)}{\omega \cdot \frac{T}{2}}$$

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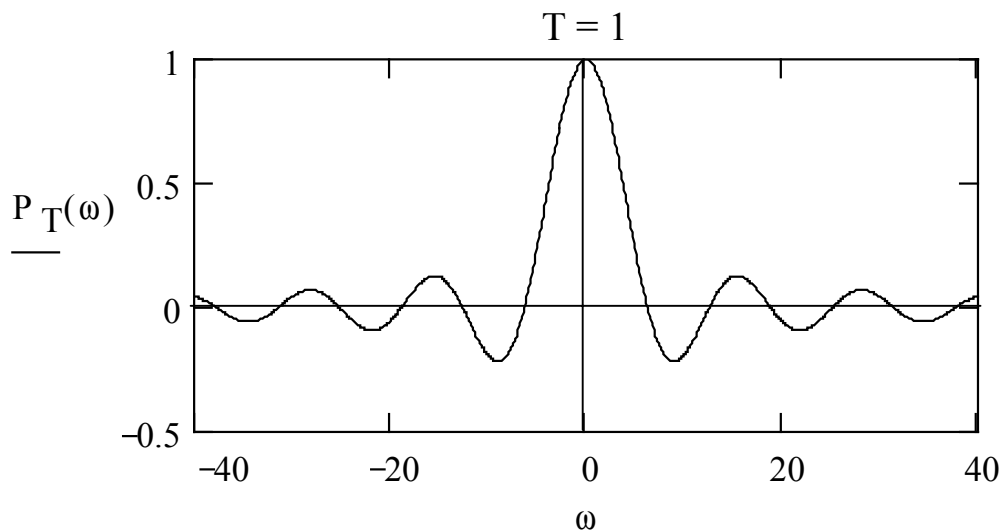


Figure 3.26

Fourier Transform of Mathematical Operations

<u>Operation</u>	$f(t)$	$F(\omega)$
Transformation	$f(t)$	$\int_{-\infty}^{\infty} f(t) \epsilon^{-j\omega t} dt$
Inversion	$\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \epsilon^{j\omega t} d\omega$	$F(\omega)$
Superposition	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(\omega) + a_2 F_2(\omega)$
Reversal	$f(-t)$	$F(-\omega)$
Symmetry	$F(t)$	$2\pi f(-\omega)$
Scaling	$f(at)$	$\frac{1}{ a } F\left(\frac{\omega}{a}\right)$
Delay	$f(t - t_0)$	$\epsilon^{-j\omega t_0} F(\omega)$
Modulation	$\epsilon^{j\omega_0 t} f(t)$	$F(\omega - \omega_0)$
Time differentiation	$\frac{d^n}{dt^n} f(t)$	$(j\omega)^n F(\omega)$
Frequency differentiation	$t^n f(t)$	$(j)^n \frac{d^n}{d\omega^n} F(\omega)$
Integration	$\int_0^t f_e(t) dt + \int_{-\infty}^t f_o(t) dt$	$\frac{1}{j\omega} F(\omega)$
Integration	$\int_{-\infty}^t f(t) dt$	$\frac{1}{j\omega} F(\omega) + \pi F(0) \delta(\omega)$
Convolution	$f_1 * f_2 = \int_{-\infty}^{\infty} f_1(\lambda) f_2(t - \lambda) d\lambda$	$F_1(\omega) F_2(\omega)$
Multiplication	$f_1(t) f_2(t)$	$\frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(\xi) F_2(\omega - \xi) d\xi$

Figure 3.27

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Properties and short table of Fourier Transforms

- If $f(t)$ is even $\implies F(\omega)$ is purely real
- If $f(t)$ is odd $\implies F(\omega)$ is purely imaginary
- The real part of $F(\omega)$ is an even function
- The imaginary part of $F(\omega)$ is an odd function

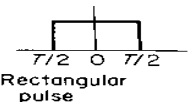
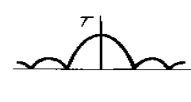
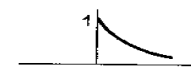
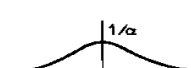
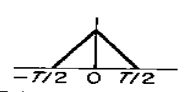
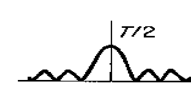
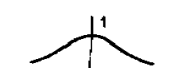
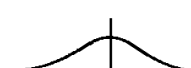
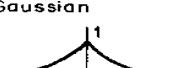

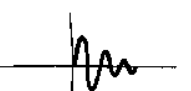
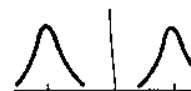
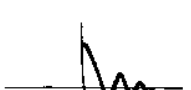





	$f(t)$	$F(\omega)$	$ F(\omega) $
Rectangular pulse	 $u(t + T/2) - u(t - T/2)$	$T \frac{\sin(\omega T/2)}{\omega T/2}$	
Exponential	 $e^{-\alpha t} u(t)$	$\frac{1}{j\omega + \alpha}$	
Triangular	 $1 - 2 \frac{ t }{T}, t < \frac{T}{2}$ 0 elsewhere	$\frac{T}{2} \left[\frac{\sin(\omega T/4)}{\omega T/4} \right]^2$	
Gaussian	 $e^{-\alpha^2 t^2}$	$\frac{\sqrt{\pi}}{\alpha} e^{-(\omega^2/4\alpha^2)}$	
Double exponential	 $e^{-a t }$	$\frac{2a}{a^2 + \omega^2}$	
Damped sine	 $e^{-at} \sin(\omega_0 t) u(t)$	$\frac{\omega_0}{(a + j\omega)^2 + \omega_0^2}$	
Damped cosine	 $e^{-at} \cos(\omega_0 t) u(t)$	$\frac{a + j\omega}{(a + j\omega)^2 + \omega_0^2}$	
	 $\frac{1}{\beta - \alpha} [e^{-\alpha t} - e^{-\beta t}] u(t)$	$\frac{1}{(j\omega + \alpha)(j\omega + \beta)}$	
Cosine pulse	 $\cos \omega_0 t [u(t + T/2) - u(t - T/2)]$	$\frac{T}{2} \left[\frac{\sin(\omega - \omega_0)T/2}{(\omega - \omega_0)T/2} - \frac{\sin(\omega + \omega_0)T/2}{(\omega + \omega_0)T/2} \right]$	

Figure 3.28 Fourier Transform of Energy Signal

ANALOG AND DIGITAL SIGNAL PROCESSING

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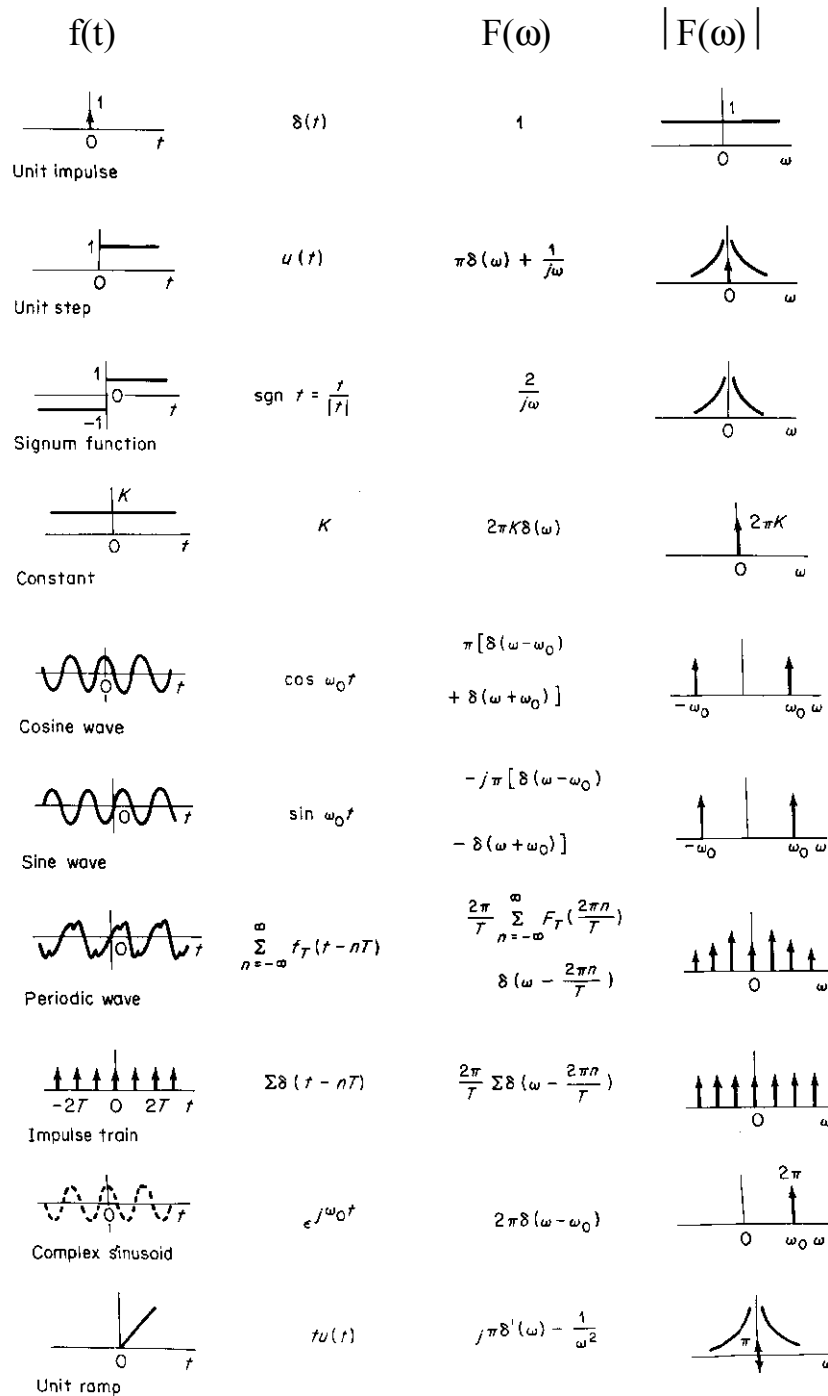


Figure 3.29 Fourier Transforms of Power Signals

Note : Since $f(t)$ does not necessary vanish with increasing t , the Fourier integral might not converge. In this case, the Fourier Transform does not exist. In order to solve this problem, $f(t)$ can be multiplied by a decaying exponential $e^{-\alpha|t|}$, with $\alpha > 0$, so that the product, $f(t) e^{-\alpha|t|}$, now vanishes with increasing or decreasing t . With this change, we end up with the definition of the *Laplace transform*.

Important result:

Impulse train in the time domain \rightarrow Impulse train in the frequency domain

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3.4.1 APPLICATIONS OF THE FOURIER TRANSFORM

Example 1

Consider the following filter :

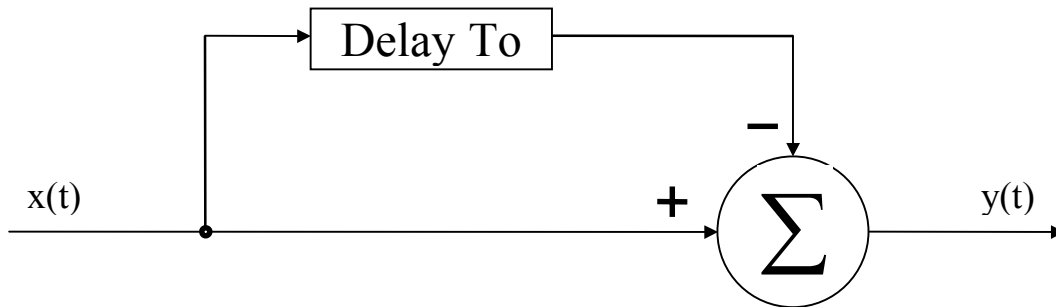


Figure 3.30

- Compute $|Y(j\omega) / X(j\omega)|$ and demonstrate that this filter is a rejector.
Apply the Fourier transform.
- Determine the smallest T_0 so that it works as a rejector on 50 Hz.
- Determine which other frequencies are also rejected and plot the magnitude versus frequency of this filter.
- Plot the impulse response and the unit step response.
- What happens if the subtractor is replaced by an adder ?

Solution :

From the Fourier transform :

$$Y(j\omega) = X(j\omega) - e^{-j\omega T_0} \cdot X(j\omega)$$

$$K(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = 1 - e^{-j\omega T_0} = 1 - \cos(\omega \cdot T_0) + j \sin(\omega \cdot T_0)$$

$$|K(j\omega)| = \sqrt{2 \cdot \sqrt{1 - \cos(\omega \cdot T_0)}}$$

a) The filter works as a rejector if :

$$1 - \cos(\omega T_0) = 0 \Rightarrow \cos(\omega T_0) = 1 \Rightarrow T_0 = 2 \pi N / \omega_{\text{rej}} \quad \text{with } N > 0 \text{ (integer)}$$

b) We get a rejection at 50 Hz and the smallest T_0 with $N = 1$.

$$\Rightarrow T_0 = 2 \pi 1 / (2 \pi 50) = 20 \text{ ms}$$

c) $\omega_{\text{rej}} = 2 \pi N / T_0 \Rightarrow \text{freq}_{\text{rej}} = N / T_0 = 50 \text{ Hz}, 100 \text{ Hz}, 150 \text{ Hz} \dots$

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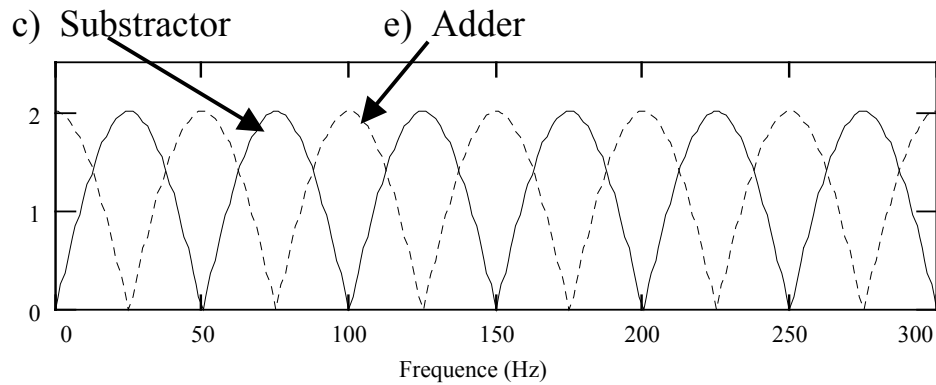
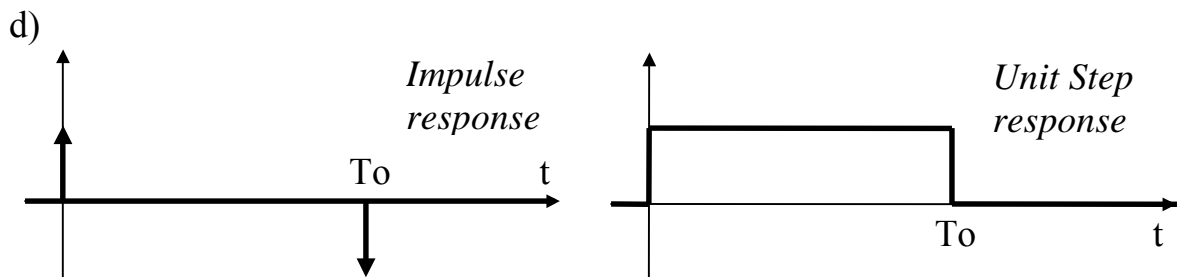


Figure 3.31

Note : With the subtractor, the DC component is also rejected.



Example 2

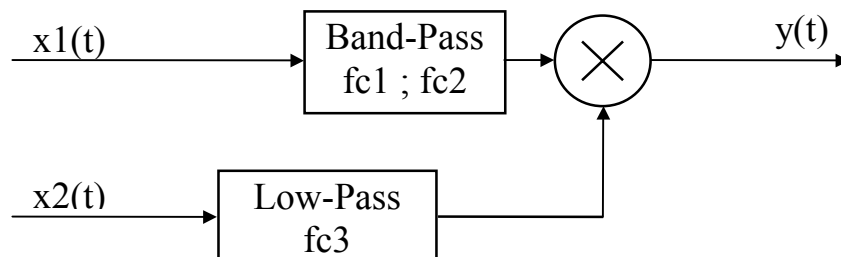


Figure 3.32

$x_1(t)$: 200 Hz, $x_2(t)$: 6000 Hz, both signals are square waves (symmetrical), 1 V pp
 $f_{c1} = 500$ Hz , $f_{c2} = 1100$ Hz , $f_{c3} = 22$ KHz

Sketch $|Y(\omega)| = f(\omega)$ in respecting the relative amplitudes.

Solution :

Reminder : $\delta(t-T) * x(t) = x(t-T)$ or $\delta(\omega - \omega_0) * X(\omega) = X(\omega - \omega_0)$

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Example 3

Let : $x_3(t) = x_1(t) \cdot x_2(t)$

$x_1(t)$ is a NRZ (non-return to zero) signal with : $T_b = 0.01$ ms

$x_2(t)$ is a 1 MHz square-wave (symmetrical).

Plot the power spectral density of x_3 .

Note : NRZ signal

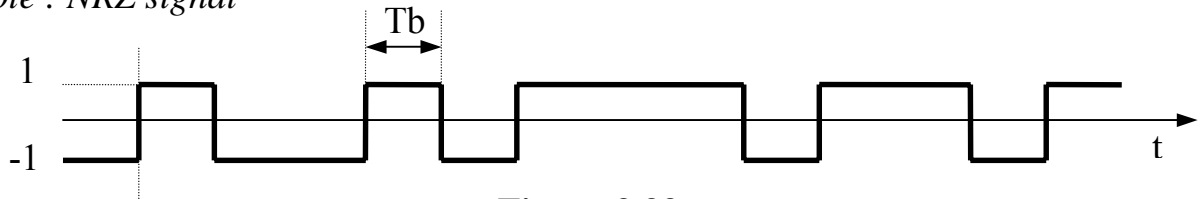


Figure 3.33

From the Wiener-Khintchine Relationship, the power spectral density of a signal is the Fourier transform of its *Auto-Correlation*.

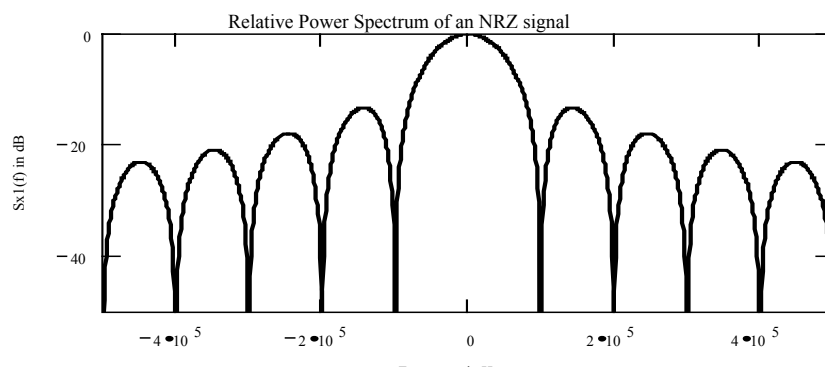


Figure 3.34

Solution : The power spectral density of the NRZ signal is convolved with the Fourier transform of the 1 MHz square-wave (odd harmonics).

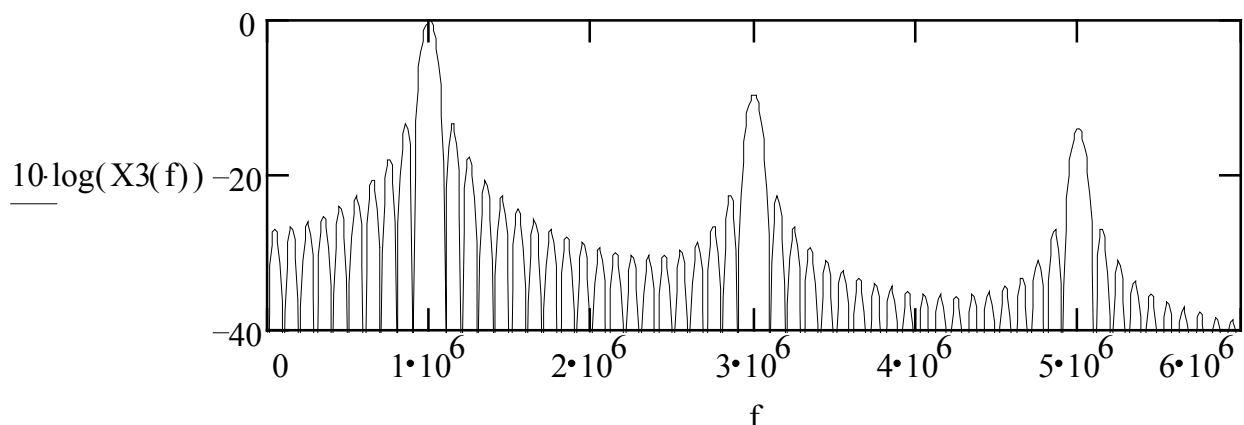


Figure 3.35

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CHAPTER 3 : LINEAR SYSTEM RESPONSE (GENERAL CASE)

3.5 LAPLACE TRANSFORM

3.5.1 INTRODUCTION

The *Laplace transform* may be considered a modification of the Fourier transform to allow the transformation of a function $f(t)$ that does not necessarily vanish with increasing t . In order to be able to solve a large variety of practical cases, $f(t)$ is multiplied by a decaying exponential $e^{-\alpha|t|}$, with $\alpha > 0$. Thus :

$$F(\alpha + j\omega) = \int_0^{\infty} f(t) \cdot e^{-(\alpha + j\omega)t} dt$$

With the substitution of the complex variable $s = \alpha + j\omega$, the familiar form of the Laplace transform is obtained :

$$F(s) = \int_0^{\infty} f(t) \cdot e^{-s \cdot t} dt$$

Thus $F(s)$, the *Laplace transform* of $f(t)$ is also the *Fourier transform* of $f(t) e^{-\alpha t}$, where α is the real part of s .

Example

A sine-wave starting at $t = 0$ (sine-wave multiplied by a unit step function at $t = 0$), does not have a Fourier transform. It does have a Laplace transform however :

$$F_{\text{sine}}(s) = \int_0^{\infty} \sin(\Omega \cdot t) \cdot e^{-s \cdot t} dt = \frac{\Omega}{s^2 + \Omega^2}$$

The following are important points in comparing the Fourier and the Laplace transforms :

1. The Laplace transform $F(s)$ is just the spectrum (Fourier transform) of $f(t) e^{-\alpha t}$, assuming that $f(t) = 0$ for $t < 0$.
2. If $f(t) = 0$ for $t < 0$ and if $F(j\omega)$ exists, then $F(j\omega)$ is equal to $F(s)$ with $s = j\omega$; i.e., with $\alpha = 0$.
3. The Fourier transform does not require that $f(t) = 0$ for $t < 0$.
4. The Laplace transformation can be performed on a wide variety of practical $f(t)$'s for which the Fourier transform does not exist.
5. In practice, the inverse transform of $F(j\omega)$ or $F(s)$ can be very tedious.

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The mathematical forms of the Laplace transform and its inverse are :

Laplace Transform	$L(f(t)) = F(s) = \int_0^{\infty} f(t) \cdot e^{-s \cdot t} dt$
Inverse Laplace Transform	$f(t) = \frac{1}{2 \cdot \pi \cdot j} \cdot \int_{\alpha - j \cdot \infty}^{\alpha + j \cdot \infty} F(s) \cdot e^{s \cdot t} ds$

3.5.2 SHORT TABLE OF TRANSFORMS

	<u>Function</u>	<u>Laplace Transform</u>	<u>Fourier Transform</u>
	f(t)	F(s)	F(jω)
Linearity	A f(t)	A F(s)	A F(j ω)
Superpo.	f(t) + g(t)	F(s) + G(s)	F(j ω) + G(j ω)
	$d f(t)/dt$	s F(s) - f(0 ⁺)	j ω F(j ω)
	$\int_{-\infty}^t f(\tau) d\tau$	F(s) / s	F(j ω) / j ω
	t f(t)	- d F(s) / ds	j d F(j ω) / d ω
Delay	$e^{-at} f(t)$, a > 0	F(s+a)	F(j ω +a)
	f(t- τ), a > 0	$e^{-\tau s} F(s)$	$e^{-j \omega \tau} F(j\omega)$
	f(t/a), a > 0	a F(as)	a F(a j ω)
Conv.	f(t) + f(-t)	-	F(j ω) + F(-j ω)
	f(t) * g(t)	F(s) G(s)	F(j ω) G(j ω)
.....			
	$\delta(t)$	1	1
	1	1/s	-
	t ⁿ	n!/s ⁿ⁺¹	-
	e^{-at}	1/(s+a)	1/(j ω +a)
	sin($\omega_0 t$)	$\omega_0 / (s^2 + \omega_0^2)$	$-j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
	cos($\omega_0 t$)	s/(s ² + ω_0^2)	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
	$e^{-at} \sin(\omega_0 t)$	$\omega_0 / [(s+a)^2 + \omega_0^2]$	$\omega_0 / [(j\omega+a)^2 + \omega_0^2]$
	$e^{-at} \cos(\omega_0 t)$	$(a+s) / [(s+a)^2 + \omega_0^2]$	$(a+j\omega) / [(j\omega+a)^2 + \omega_0^2]$

The Transforms assume that f(t) = 0 for t < 0
 (excepted the Fourier Transform of sin($\omega_0 t$) and cos($\omega_0 t$))

Note: $A e^{-at} \sin(\omega_0 t) + B e^{-at} \cos(\omega_0 t) = C e^{-at} \cos(\omega_0 t + \varphi)$

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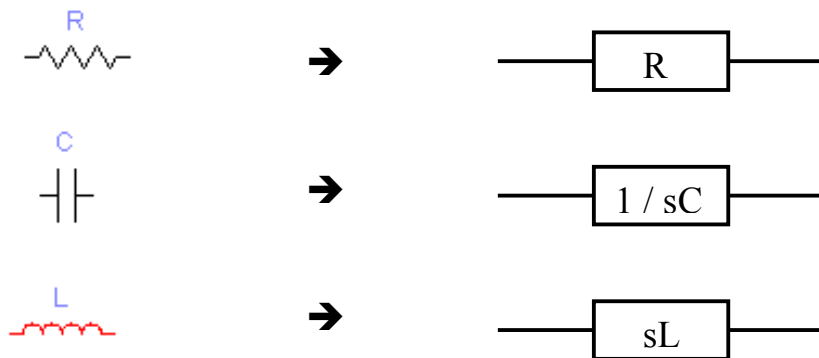
3.5.3 APPLICATIONS OF THE LAPLACE TRANSFORM

Example 1 : Network analysis (Transient analysis)

1) Resistor : $u_R(t) = R i_R(t) \rightarrow U_R(s) = R I_R(s)$

2) Capacitor : $i_c(t) = C du_c/dt \rightarrow I_c(s) = C [s U_c(s) - u_c(0^+)]$

3) Inductor : $u_L(t) = L di_L(t)/dt \rightarrow U_L(s) = L [s I_L(s) - i_L(0^+)]$



In the case of the R-C low-pass filter :

$$U_o(s) = U_i(s) \cdot \frac{\frac{1}{s \cdot C}}{R + \frac{1}{s \cdot C}} = U_i(s) \cdot \frac{1}{1 + s \cdot R \cdot C}$$

If $u_i(t)$ is a unit step, then : $u_o(t) = L^{-1}[U_i(s) \cdot 1/(1+s R C)]$. From the *partial fraction expansion* techniques :

$$\frac{1}{s \cdot (1 + s \cdot R \cdot C)} = \frac{A}{s} + \frac{B}{1 + s \cdot R \cdot C} = \frac{1}{s} - \frac{R \cdot C}{1 + s \cdot R \cdot C} = \frac{1}{s} - \frac{1}{s + \frac{1}{R \cdot C}}$$

Inverse Laplace transform : $u_o(t) = u(t) - e^{-\frac{1}{R \cdot C} \cdot t} \cdot u(t)$

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Example 2

Suppose that all the initial conditions of the following circuit are zero. Determine $i(t)$.

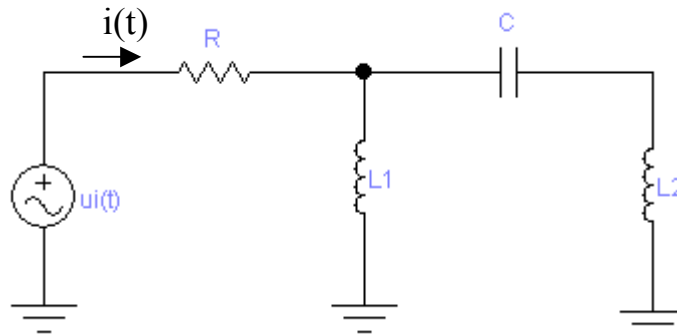


Figure 3.36

$$Z_{eq}(s) = R + Z_{L1}(s) // (Z_C(s) + Z_{L2}(s)) =$$

$$R + \frac{s \cdot L1 \cdot \left(\frac{1}{s \cdot C} + s \cdot L2 \right)}{s \cdot L1 + \frac{1}{s \cdot C} + s \cdot L2} = \frac{R \cdot (s^2 \cdot L1 \cdot C + 1 + s^2 \cdot L2 \cdot C) + s \cdot L1 \cdot (1 + s^2 \cdot L2 \cdot C)}{s^2 \cdot (L1 \cdot C + L2 \cdot C) + 1} = \frac{s^3 \cdot L1 \cdot L2 \cdot C + s^2 \cdot (R \cdot L1 \cdot C + R \cdot L2 \cdot C) + s \cdot L1 + R}{s^2 \cdot (L1 \cdot C + L2 \cdot C) + 1}$$

$$i(t) = \mathcal{L}^{-1} [U_i(s) / Z_{eq}(s)]$$

Assuming that $u_i(t)$ is limited to one cycle of a sine-wave of period T :

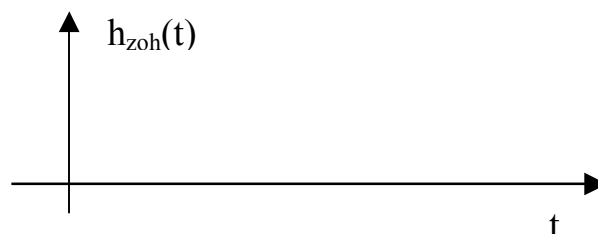
$$u_i(t) = A \sin(\omega_0 t) u(t) - A \sin(\omega_0 t) u(t-T) \rightarrow U_i(s) = A \omega_0 / (s^2 + \omega_0^2) (1 - e^{-sT})$$

Note: The inverse Laplace transform can be very difficult to perform. **However, if $u_i(t)$ is periodic, then it can be shown that the steady state solution (once the transients have died down) can be simply computed in replacing s by $j\omega$ and in applying the harmonic analysis techniques (section 2.3).**

Example 3

What is the Laplace transform of the zero-order hold (zoh) ?

$$h_{zoh}(t) = u(t) - u(t-T)$$



$$\rightarrow H_{zoh}(s) = (1/s) - e^{-Ts}/s = (1 - e^{-Ts})/s.$$

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CHAPTER 3 : LINEAR SYSTEM RESPONSE (GENERAL CASE)

3.5.4 POLE-ZERO AND STABILITY

A transfer function $H(s)$ can almost always be written as a rational function, i.e., as a ratio of polynomials :

$$H(s) = \frac{B_0 + B_1 \cdot s + B_2 \cdot s^2 + \dots + B_n \cdot s^n}{A_0 + A_1 \cdot s + A_2 \cdot s^2 + \dots + A_m \cdot s^m}$$

In principle the polynomials can always be factored, and $H(s)$ can be written in the pole-zero form ($n \leq m$) :

$$H(s) = C \cdot \frac{(s - q_1) \cdot (s - q_2) \cdot \dots \cdot (s - q_n)}{(s - p_1) \cdot (s - p_2) \cdot \dots \cdot (s - p_m)}$$

The constant C in front of the latter ratio is equal to B_n/A_m , and the p 's and q 's depend on the A 's and B 's respectively. The roots of the numerator, i.e., q_1, \dots, q_n are the *zeros* of $H(s)$, and p_1, \dots, p_m are the *poles* of $H(s)$. The poles and the zeros of $H(s)$ may be real, complex, or zero, as they may be for any polynomial function.

A plot of the poles and zeros of $H(s)$ on the complex s -plane provides a geometric interpretation of the amplitude response and the phase shift, which are defined as :

With $s \rightarrow j\omega$, $H(j\omega) = R(\omega) + j I(\omega)$ and

$$\begin{aligned} \text{amplitude response} &= [R^2(\omega) + I^2(\omega)]^{1/2} = |H(j\omega)| \\ \text{phase shift} &= \tan^{-1}[I(\omega) / R(\omega)] \end{aligned}$$

Given $H(j\omega)$ in the pole-zero form, the amplitude response must be

$$|H(j\omega)| = C \cdot \frac{|j\omega - q_1| \cdot |j\omega - q_2| \cdot \dots \cdot |j\omega - q_n|}{|j\omega - p_1| \cdot |j\omega - p_2| \cdot \dots \cdot |j\omega - p_m|}$$

and each component of the form $|j\omega - x|$ can be interpreted as a distance on the s -plane from the point given by ω on the imaginary axis to the point x . Thus the amplitude response becomes a ratio of products of distances on the s -plane. Similarly, the total phase shift becomes an algebraic sum of angles. These properties are best illustrated by an example.

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Example (Pole-zero plot)

Plot the poles and zeros of

$$H(s) = \frac{C \cdot s}{s^2 + 2 \cdot s + 5} = \frac{C \cdot s}{(s + 1 + j \cdot 2) \cdot (s + 1 - j \cdot 2)}$$

This transfer function has a single zero at $s = 0$ and poles at $s = -1 \pm j 2$.

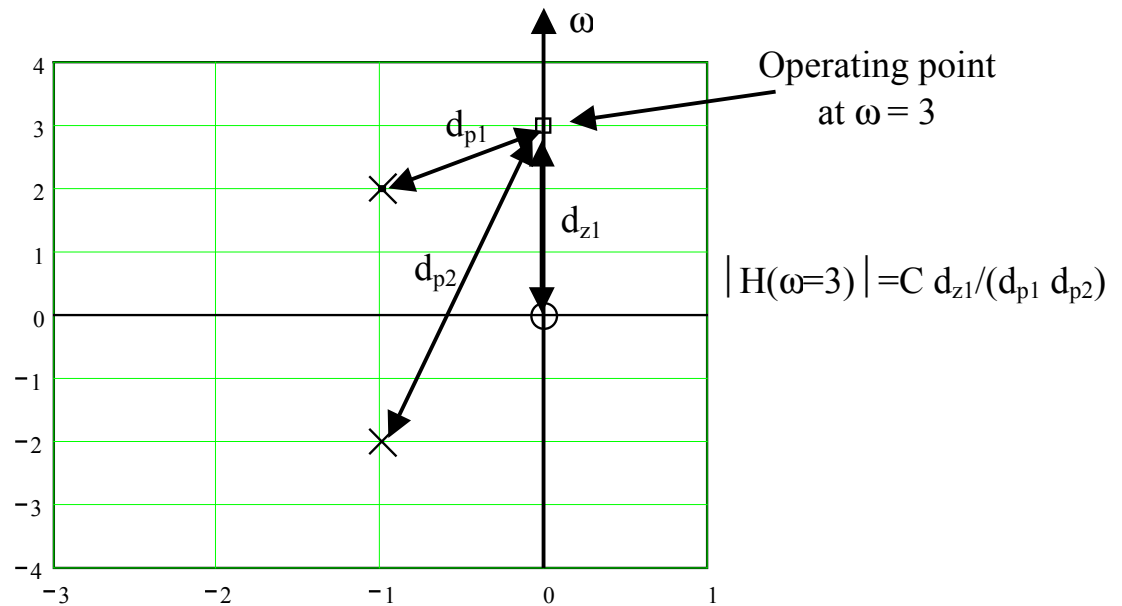


Figure 3.37 Pole-zero plot (s-plane)

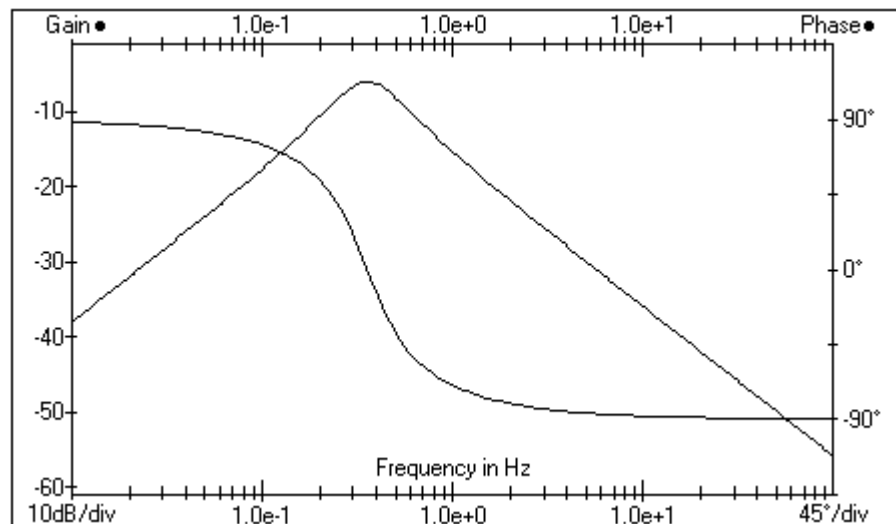


Figure 3.38 Bode Plot, Gain and phase shift (in control theory, ω , is in rad/s)

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The s-plane plot is useful in the design as well as in the analysis of linear systems. In the design process, one thinks in terms of placing the poles and zeros on the s-plane to achieve certain objectives, with $H(s)$ then being found from the pole-zero form. In connection with this, the s-plane plot has several properties worth noting :

- 1) Poles and zeros must be real or appear in conjugate pairs.
- 2) A pole (zero) on the $j\omega$ -axis implies that $|H(j\omega)|$ is infinite (zero) at some specific frequency.
- 3) A pole in the right half plane causes instability in the sense that the response to a transient input signal will increase rather than decay.
- 4) Poles not on the real axis in general cause oscillation in the output.

The instability suggested by property 3 can be demonstrated by expanding $H(s)$ into a sum of fractions :

$$H(s) = \frac{C_1}{s - p_1} + \frac{C_2}{s - p_2} + \dots + \frac{C_m}{s - p_m}$$

The inverse Laplace transform of each individual fraction is an exponential of the form : $C_i e^{p_i t}$

If one of the poles, say $p_n = \alpha_n + j\omega_n$, lies in the right-half plane so that α_n is positive, then its contribution to the impulse response $e^{(\alpha_n + j\omega_n)t}$, will increase without bound, in effect causing the transient response to be unstable.

Example (s-plane filter design with SystemView)

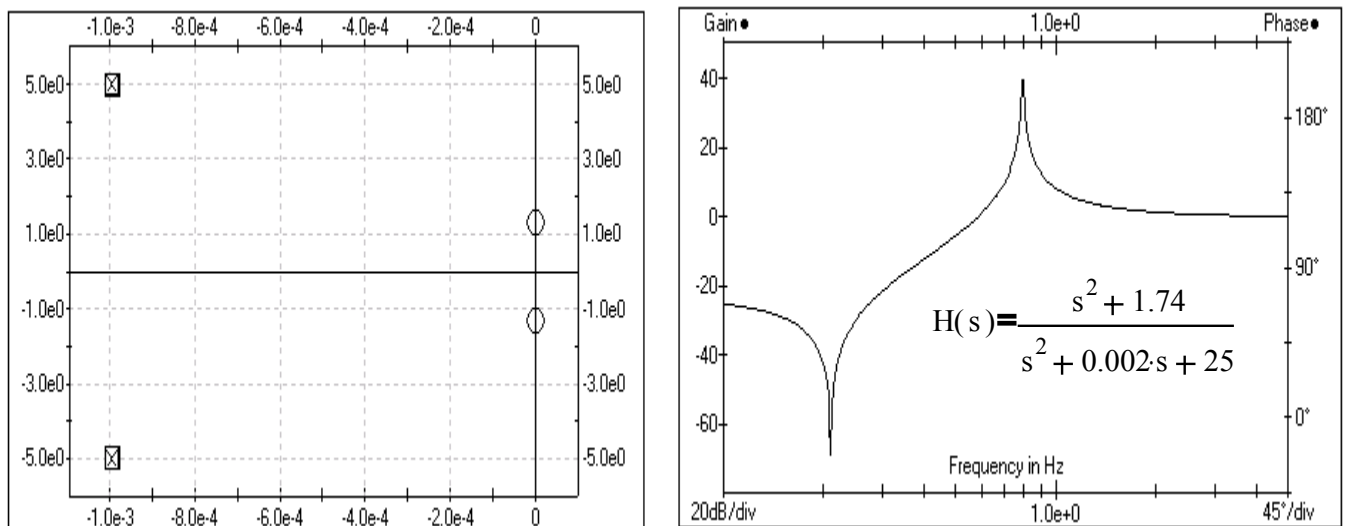


Figure 3.39

zeros : $0 - j 1.32$, $0 + j 1.32$ on the $j\omega$ -axis \rightarrow rejector filter

poles : $10^{-3} - j 5$, $10^{-3} + j 5$ close from the $j\omega$ -axis \rightarrow narrow BP filter

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Example of literal computation (MathCad)

Consider the second-order low-pass filter of Fig. 3.8. Compute, literally, its unit step response.

1) Transfer function in the Fourier domain :

$$H(j\omega) = \frac{\frac{1}{j \cdot \omega \cdot C}}{R + j \cdot \omega \cdot L + \frac{1}{j \cdot \omega \cdot C}}$$

2) Transfer function in the Laplace domain ($j\omega \rightarrow s$) :

$$H(s) = \frac{\frac{1}{s \cdot C}}{R + s \cdot L + \frac{1}{s \cdot C}} = \frac{1}{1 + s \cdot C \cdot R + s^2 \cdot L \cdot C}$$

3) $Y(s) = H(s) X(s)$ with $X(s) = 1/s$ (Laplace transform of the unit step signal)

$$Y(s) = \frac{1}{s} \cdot \frac{1}{1 + s \cdot C \cdot R + s^2 \cdot L \cdot C}$$

4) $y(t) = \mathcal{L}^{-1} [Y(s)]$ (MathCad Symbolic Inverse Transform)

$$y(t) = 1 - R \cdot \exp\left(\frac{-1}{2} \cdot \frac{R}{L} \cdot t\right) \cdot \sin\left[\frac{1}{2} \cdot \frac{\sqrt{4 \cdot L - R^2 \cdot C}}{(\sqrt{C \cdot L})} \cdot t\right] \cdot \frac{\sqrt{C}}{\sqrt{4 \cdot L - R^2 \cdot C}} - \exp\left(\frac{-1}{2} \cdot \frac{R}{L} \cdot t\right) \cdot \cos\left[\frac{1}{2} \cdot \frac{\sqrt{4 \cdot L - R^2 \cdot C}}{(\sqrt{C \cdot L})} \cdot t\right]$$

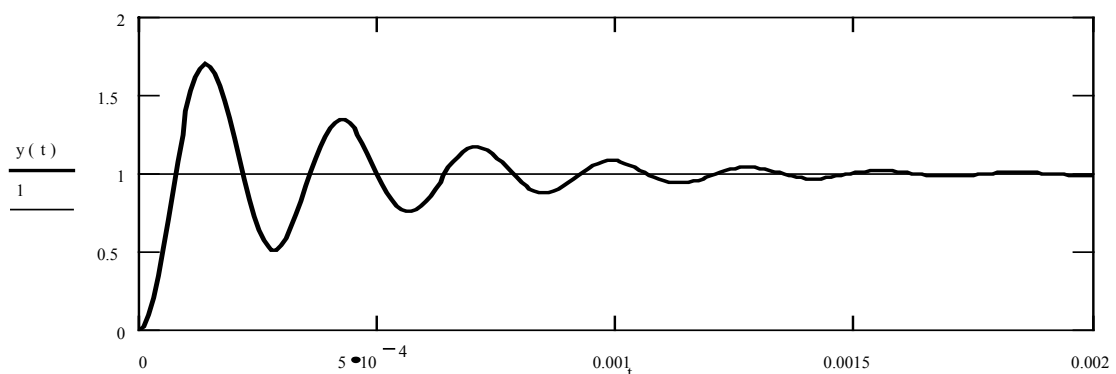
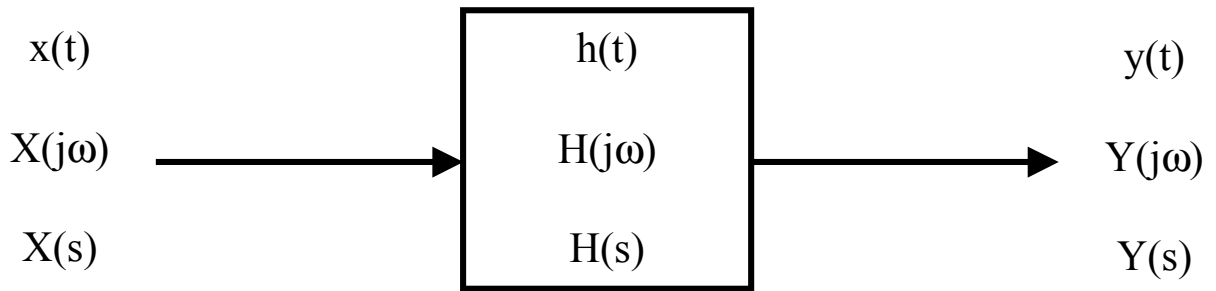


Figure 3.40

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3.6 SUMMARY



$h(t)$: Impulse response of the system; it can be determined experimentally with an input signal $x(t)$ approximating an impulse response (p. 3.4).

$$h(t) = \mathbf{F}^{-1} [H(j\omega)] \quad \text{or} \quad h(t) = \mathbf{L}^{-1} [H(s)]$$

$H(j\omega)$: Fourier transform of $h(t)$, $H(j\omega) = \mathbf{F}[h(t)]$, gives the amplitude and phase response of the system or its power spectrum.

$H(s)$: Laplace transform of $h(t)$, $H(s) = \mathbf{L}[h(t)]$

Convolution: $y(t) = x(t) * h(t) = \mathbf{F}^{-1} [Y(j\omega)]_{\text{if it exists}} = \mathbf{L}^{-1} [Y(s)]$

$$Y(j\omega) = X(j\omega) H(j\omega) = \mathbf{F} [y(t)]$$

$$Y(s) = X(s) H(s) = \mathbf{L} [y(t)]$$

In harmonic analysis (periodic input signal), we can go from the Laplace transform to the Fourier transform with $s \rightarrow j\omega$

If a transfer function is given in terms of $j\omega$ ($j\omega$)² .., its Laplace transform is simply given in replacing $j\omega$ by s ($j\omega \rightarrow s$).

ANALOG AND DIGITAL SIGNAL PROCESSING

CHAPTER 3 : LINEAR SYSTEM RESPONSE (GENERAL CASE)

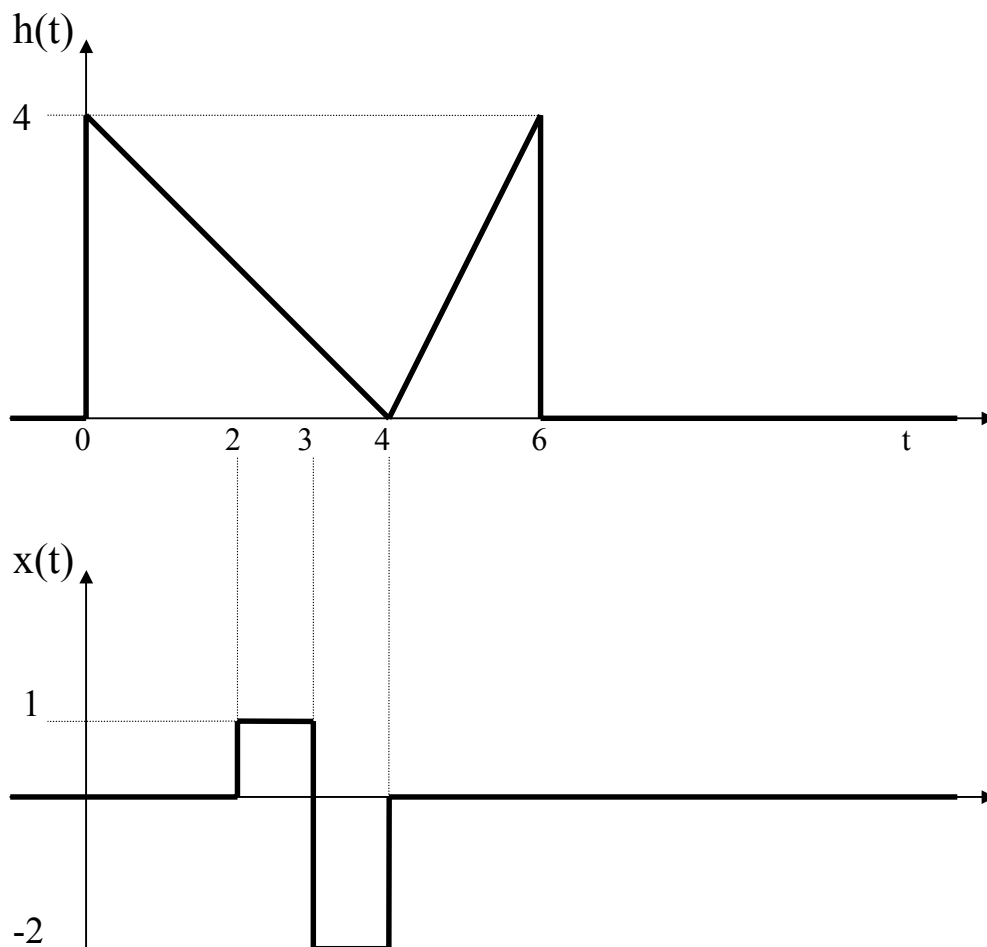
3.7 PROBLEMS

Problem 3.1

- Write the equation of a "chirp" if $f_{\min} = 10$ KHz, $f_{\max} = 50$ KHz and the sweep time $t_{\text{sweep}} = 2$ ms.
- Repeat a) if the frequency interval ($f_{\max} - f_{\min}$) is logarithmically swept.
- If a "chirp" is used to quickly determine the magnitude versus frequency of a given filter, how do you approximately determine t_{sweep} ? (what influences its optimum choice ?).

Problem 3.2

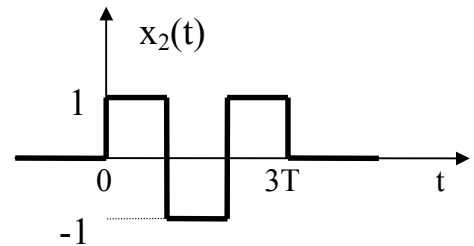
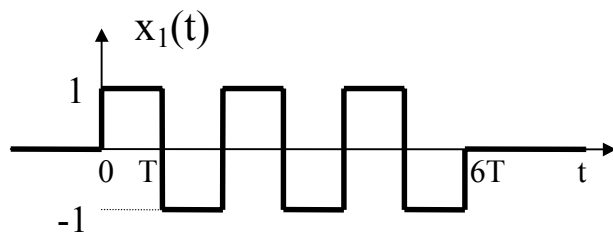
- Sketch $y(t)$ if $y(t) = x(t) * h(t)$. Refer to the approach presented in section 3.3.
- Repeat a) with MathCad or any other mathematical software. Show the effect of the sampling rate (which is the inverse of the time between two consecutive samples) on the accuracy of the result.



ANALOG AND DIGITAL SIGNAL PROCESSING

CHAPTER 3 : LINEAR SYSTEM RESPONSE (GENERAL CASE)

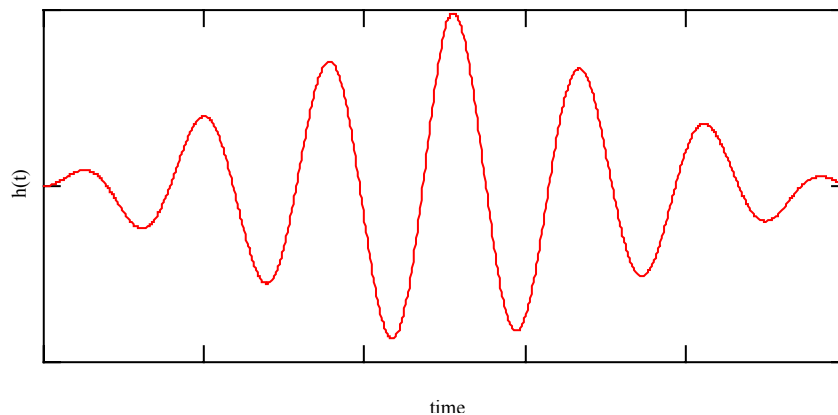
Problem 3.3



- Sketch $y(t) = x_1(t) * x_2(t)$.
- Compute y_{\max} and indicate the corresponding time.

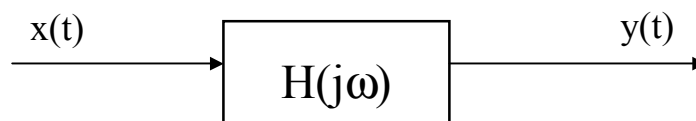
Problem 3.4

The impulse response $h(t)$ of a filter is the following :



With the input signal $x(t)$ been identical to $h(t)$, sketch $y(t)$.

Problem 3.5



$$x(t) = U(t-\tau_1) - U(t-2\tau_1) \quad \text{and} \quad H(j\omega) = 1 / (1 + j\omega\tau_2)$$

Give two different approaches and their respective mathematical expressions to compute $y(t)$. *Do not solve!*

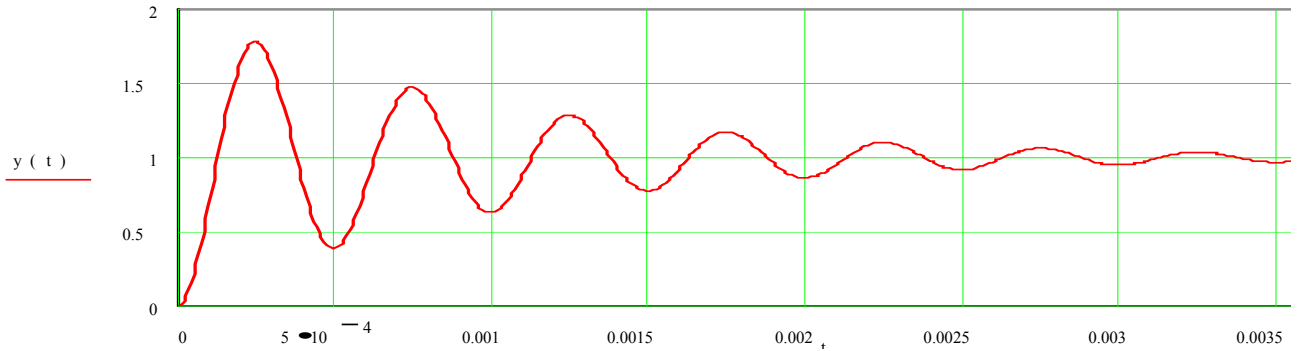
Sketch $y(t)$ with $\tau_1 = 1\text{ms}$ and $\tau_2 = 5\text{ms}$ (Hint: Step function)

ANALOG AND DIGITAL SIGNAL PROCESSING

CHAPTER 3 : LINEAR SYSTEM RESPONSE (GENERAL CASE)

Problem 3.6

This is the "step response" of a linear filter :



The following function models this signal :

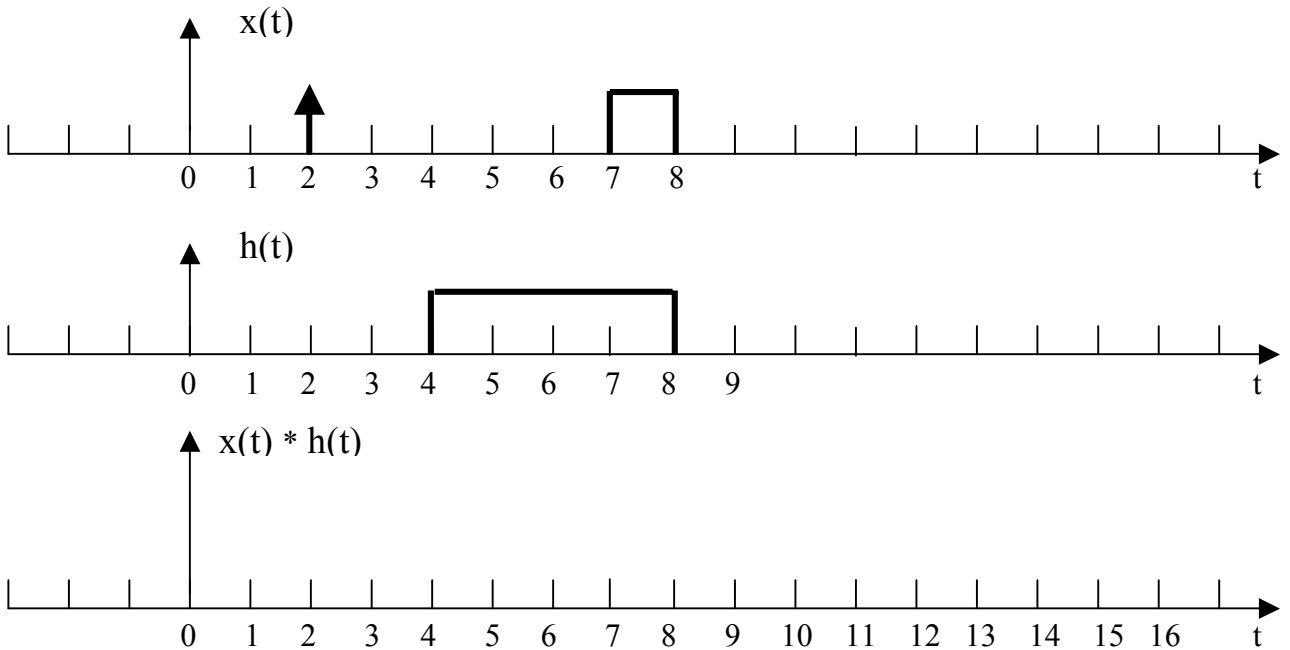
$$y(t) := A + B \cdot e^{-C \cdot t} \cdot \cos(\omega_0 \cdot t)$$

Determine all its parameters, i.e. $A =$, $B =$, $C =$, $\omega_0 =$.

What type of filter produces a "step response" like that ?

Problem 3.7

Convolve $x(t)$ with $h(t)$

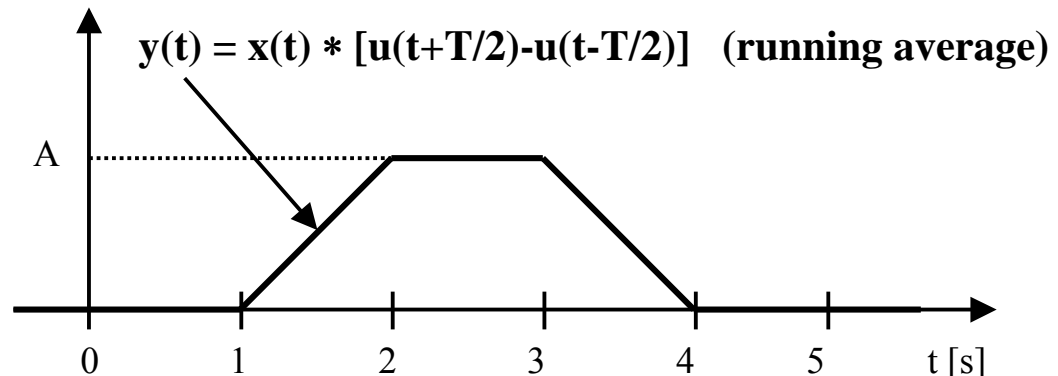


ANALOG AND DIGITAL SIGNAL PROCESSING

CHAPTER 3 : LINEAR SYSTEM RESPONSE (GENERAL CASE)

Problem 3.8 (Decorrelation)

Find $x(t)$ with $T = 1$ s.

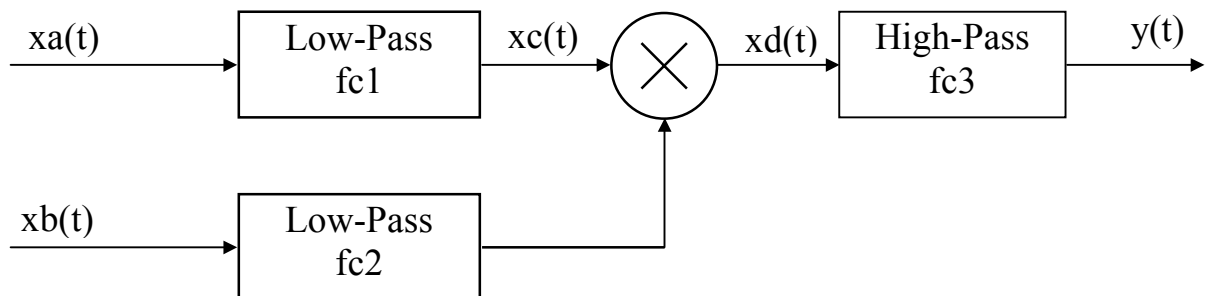


Problem 3.9

An analog filter has the following transfer function : $H(j\omega) = (1 - j\omega \tau) / (1 + j\omega \tau)$.

- What kind of filter is it ?
- Plot the amplitude and the phase characteristics of this filter.
- Suggest an application of this filter.

Problem 3.10



with : $x_a(t)$: 1 KHz square-wave (symmetrical) , 1 V peak-to-peak
 $x_b(t)$: 10 KHz square-wave (symmetrical) , 1 V peak-to-peak
 $f_{c1} = 4$ KHz , $f_{c2} = 40$ KHz , $f_{c3} = 20$ KHz

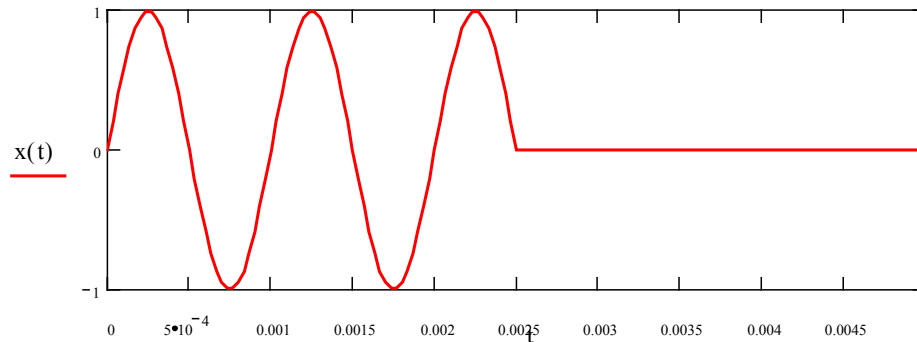
- What is the equation of $x_c(t)$?
- Draw the power spectrum of y , $|Y(f)|$, in respecting the relative amplitudes and indicate clearly the frequencies of interest.

ANALOG AND DIGITAL SIGNAL PROCESSING

CHAPTER 3 : LINEAR SYSTEM RESPONSE (GENERAL CASE)

Problem 3.11

The input signal $x(t)$ of a R-C low-pass filter is the following :

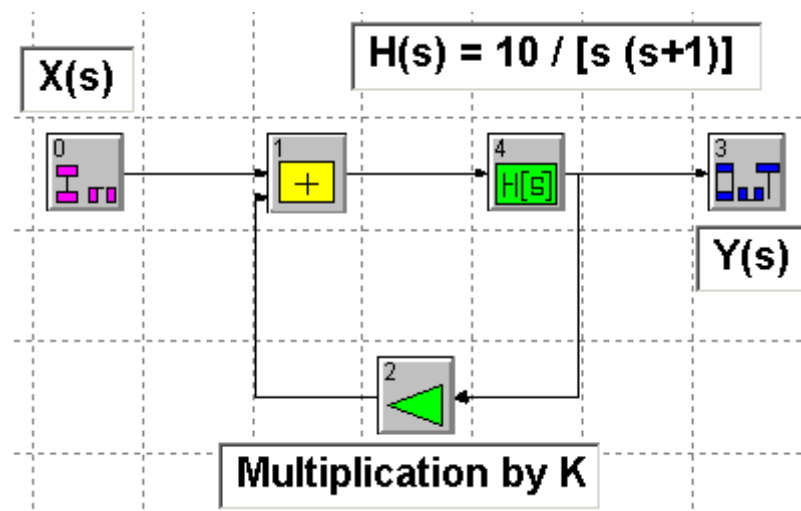


Determine $y(t)$ literally (compute the inverse Laplace transform with MathCad, Mathematica) and plot the following cases :

- a) $\tau = 0.1$ ms and b) $\tau = 1$ ms

Problem 3.12

- a) Determine the transfer function of the system shown below.
b) Find the conditions on K guarantying the stability.



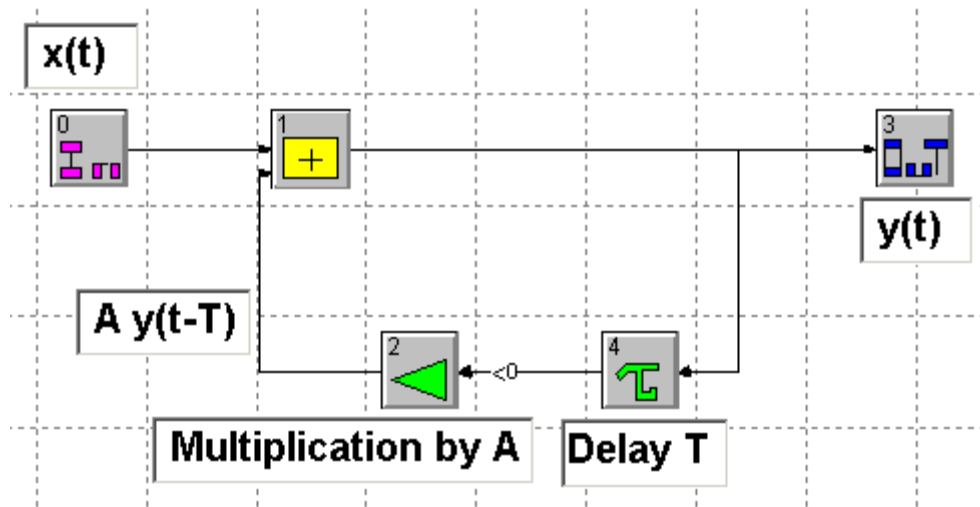
- c) Compute and plot the impulse response with $K = -2$
d) Compute and plot the amplitude response of this system.

ANALOG AND DIGITAL SIGNAL PROCESSING

CHAPTER 3 : LINEAR SYSTEM RESPONSE (GENERAL CASE)

Problem 3.13

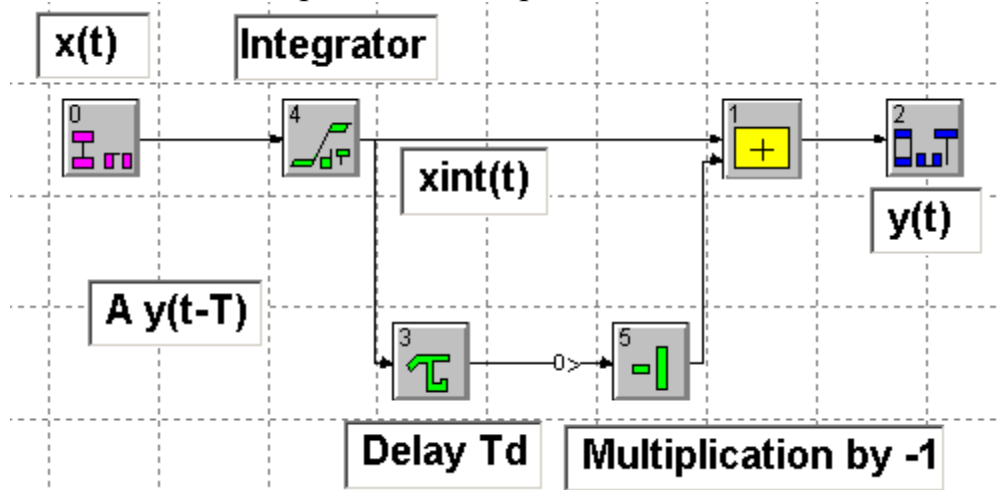
Consider the following circuit :



- Compute $K(s) = Y(s)/X(s)$.
- With $T = 20$ ms and $A = 0.99$, sketch $|K(j\omega)|$.
- If $A = 0.9$ and $T = 20$ ms, plot the impulse response.

Problem 3.14

- Prove that this circuit is a “Running Averager”.
- Test its behavior with a practical example.



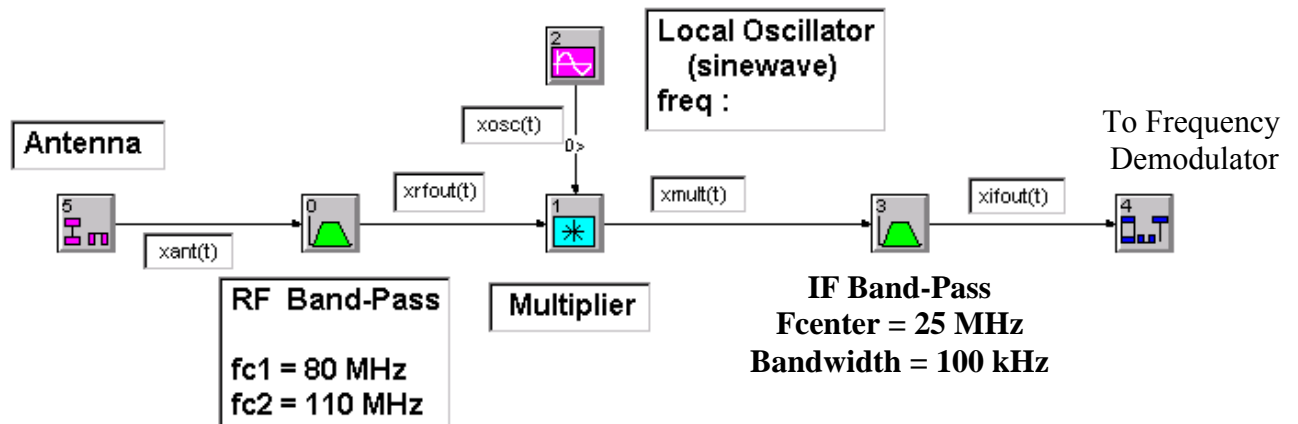
- What is its transfer function ? $|Y(j\omega) / X(j\omega)| = f(\omega)$

ANALOG AND DIGITAL SIGNAL PROCESSING

CHAPTER 3 : LINEAR SYSTEM RESPONSE (GENERAL CASE)

Problem 3.15 (Convolution in the frequency domain)

The bloc diagram of a FM receiver is the following :



- What local oscillator frequency do you choose if you wish to receive a FM station that transmits on 100 MHz ?
- What are the spectral characteristics (magnitude) at the various receiver locations if you assume that the FM spectrum available (i.e. 80 MHz \rightarrow 110 MHz) is fully occupied with stations picked-up with equal power ?
- What is the advantage of this structure compare to using a “narrow” RF band-pass filter (no multiplier, no local oscillator) ?
- What happens if the local oscillator generates square-waves instead of sine-waves ?

Problem 3.15 (Algebra)

Given: $A e^{-at} \sin(\omega_0 t) + B e^{-at} \cos(\omega_0 t) = C e^{-at} \cos(\omega_0 t + \varphi)$

If A and B are known, find C and φ