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TIME–FREQUENCY CHARACTERISTICS OF NON-LINEAR SYSTEMS

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The time-varying characteristics of non-linear systems responding to typical identification signals are addressed and the potential for identifying some time-varying patterns is shown. Specifically, the instantaneous frequency of system responses is analysed, and shown to characterise non-linear behaviour. The analysis is done via the Hilbert transform. Examples include a Duffing oscillator and a memoryless system of the type $y = x^3$. The possibility of enhancing the time–frequency resolution, as compared to classic Fourier methods, is also briefly addressed.

1. INTRODUCTION

1.1. SOME GENERAL OBSERVATIONS

The last decade has witnessed a sudden increase in the interest in time–frequency analysis methods by the engineering community, particularly in the area of diagnostics e.g. [1, 2].

Frequency domain representation is traditionally computed via Fourier methods (although model based methods can also be used for this purpose). Fourier series and transforms cannot easily localize information in the time domain. This limitation prompted the development of combined time–frequency methods many decades ago. However, the works of Gabor [3], Priestley [4], and others were virtually ignored by practitioners (certainly in the area of diagnostics), and only spectrograms, waterfall representations, with their inherent limitation, were used in an almost intuitive fashion. The situation has changed significantly in the last decade and a recent review paper [5] addresses some of the issues involved.

1.2. SIGNALS AND SYSTEMS

Limiting ourselves to the area of mechanical engineering, most of the works being published now show the application of time–frequency methods in analysing, describing and classifying signals. Typically the response of system being monitored is analysed in order to show the potential for recognising the existence of some phenomena, for example an existing or developing defect. Localisation both in time and frequency can enhance the identification of many defects. Signal processing often allows us to 'look' at information from different perspectives, and in that sense, time–frequency analysis is another important tool for such purposes.

In addition to the representation of system responses, signals can also represent systems. Typically the impulse response of a linear system completely describes its properties, and can be used to model it. For such a system, the response $y(t)$ is given as a convolution

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of an excitation $x(t)$ and the impulse response $h(t)$: $y(t) = h(t) * x(t)$, where $*$ denotes convolution. The convolution being commutative, interpretation of the excitation signal and the system's signal (i.e. the impulse response) as being essentially different is not justified, and the same signal processing tools can be applied to both. It is thus natural to inquire whether time–frequency methods might be useful for analysing signals describing systems, for example impulse responses. Let us look at the one corresponding to a single degree of freedom (sdof) second-order system, typically a decaying oscillating response [Fig. 1(a)]. For a linear system the oscillating frequency is constant, while the intensity (envelope) is well described in the time domain. No additional insight should be gained by 'looking' at the information in the time–frequency domain, as opposed to the time or frequency domain separately.

1.3. NON-LINEAR SYSTEMS

Consider the impulse response of an sdof second-order non-linear system, for example a Duffing system with a cubic stiffening term

$$m\ddot{y} + c\dot{y} + k(1 + \varepsilon y^2)y = 0. \tag{1}$$

The impulse response of the system is shown in Fig. 1(b). The period of oscillation varies during the decay. The physical interpretation of this behaviour is easily obtained by observing that the increased stiffness for large amplitude excursions will be accompanied by a faster oscillation. This behaviour could be emphasised via a time–frequency representation, showing variations of intensities and frequencies along the time axis. One traditional way to represent this has been via 'backbone curves' [6], showing the average natural frequency as a function of oscillation amplitude (this is shown in Fig. 4 for the Duffing oscillator). Thus, it seems that time–frequency methods could be an important analysis tool for non-linear systems, particularly signals representing systems, for example impulse responses. Having started from the premise that the interpretation/analysis of signals and impulse responses could be interchanged in the case of linear systems (due to convolution properties), and thus wondering whether time–frequency methods should be applied to both, we find ourselves suggesting their application to non-linear systems where the convolution operation does not apply!

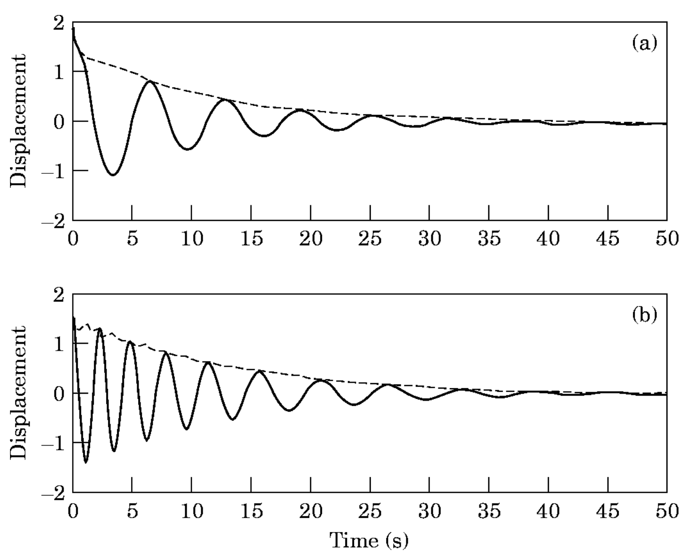

Figure 1. Free responses: (a) linear vibration system; (b) the Duffing model ($\varepsilon = 5$).

2. THE HILBERT TRANSFORM AS AN ANALYSIS TOOL

The Hilbert transform is defined by

$$H[y(t)] = \tilde{y}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y(\tau)}{t-\tau}\,\mathrm{d}\tau = y(t) * \frac{1}{\pi t}. \tag{2}$$

The Hilbert transform of an harmonic signal is also harmonic and for $x = x_{\max}\cos(t)$, $H[x] = x_{\max}\sin(t)$, and $H[H(x(t)] = -x(t)$. Thus the Hilbert transform is often interpreted as a 90° phase shifter. The transform forms the basis of the definition of an analytic signal. This is the natural extension of real signals to complex signals which is one of the cornerstones in the discipline of signal processing. The transform enables us to define signals using a complex exponential, as

$$X(t) = x(t) + j\tilde{x}(t) = A(t)\,\mathrm{e}^{j\psi(t)} \tag{3}$$

with $A(t) = [y^2(t) + \tilde{y}^2(t)]^{-1/2}$, $\psi(t) = \tan^{-1}\tilde{y}(t)/y(t)$. Such a representation has been found useful for many types of signal, especially narrowband ones, where $A(t)$ is usually 'slow' compared to the signals temporal variations; hence the names of envelope and instantaneous phase for $A(t)$ and $\psi(t)$. The definition of instantaneous frequency as the time derivative of $\psi(t)$: $f(t) = \mathrm{d}\psi/\mathrm{d}t$ can form the basis of a time–frequency representation, see for example [5]. This is especially convenient for situations where the product of time duration and signal bandwidth is sufficiently large. Such a situation exists, for example, for signals of oscillating character. However, problems arise with multicomponents signals, for example chirps (swept sines) whose frequencies vary differently [7]. In what follows the time–frequency patterns associated with non-linear systems are investigated, where the time functions are of relatively regular oscillating character. For these types of signal, analysis via the Hilbert transforms show considerable potential.

3. SYSTEMS WITH MEMORY

We shall consider systems described by a differential equation and limit ourselves to the free response of an sdof second-order non-linear systems. A typical conservative non-linear system would be of the form

$$\ddot{y} + k(y) = 0. \tag{4}$$

For a mass spring system, the second term would represent the restoring force per unit mass as a function of the displacement y. Traditionally the motion is studied in the phase plane y, $\dot{y}$, and an average natural period is then computed as [6]

$$T_{\text{average}} = 4\int_0^{y_{\max}} \frac{\mathrm{d}y}{\left[2\int_y^{y_{\max}} k(\xi)\,\mathrm{d}\xi\right]^{1/2}}. \tag{5}$$

The average period T_{average} and its reciprocal $f_{\text{average}} = 1/T_{\text{average}}$ are thus a function of $y_{\max}$. We now define a time-varying frequency as the reciprocal of the time-varying period T, computed by

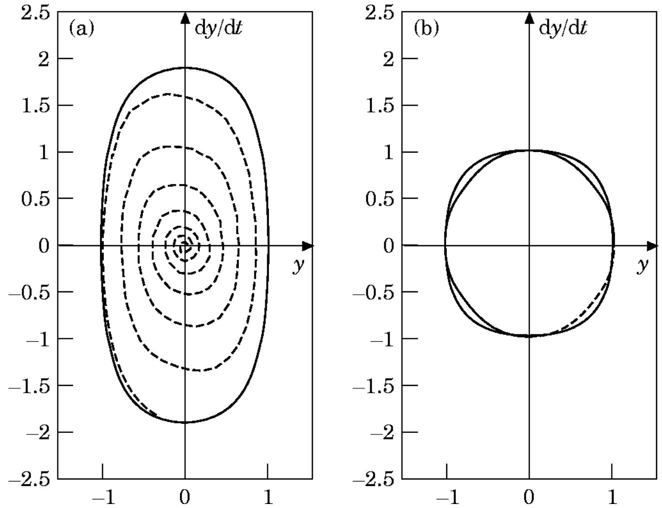

Figure 2. Duffing model ($\varepsilon = 5$): (a) phase plane (——, $c=0$, ---, $c = 0.1$); (b) relative phase plane (——), and analytic signal (---) representation.

$$T = 4 \int_{l_u}^{l_u} \frac{\mathrm{d}u}{\left[2 \int_{y}^{y\,\max} k(\xi)\,\mathrm{d}\xi \right]^{1/2}} \tag{6}$$

where the complete cycle has been divided into n sections $\Delta\varphi = \pi/2n$, the new limits $l_1 = \cos[\pi/2 - \Delta\varphi(i-1)]$, $l_u = \cos(\pi/2 - \Delta\varphi i)$, and T becoming continuous in the limit for large n. As an example we show the phase plane for a Duffing oscillator (Fig. 2). Figure 2(a), corresponding to the specific case of $m = 1$, $c = 0$, $\varrho = 5$, shows the phase plane, and Fig. 3(a) and (b) the time variations of the period and corresponding displacements. Thus, the system is characterised by a time-varying natural frequency $1/T(t)$.

We now attempt to analyse the system via the Hilbert transform. This enables us to compute the time varying envelope $A(t)$ and the instantaneous phase $\psi(t)$ and frequency (the first derivative of the phase). We may plot a radius vector $A(t)$ with phase $\psi(t)$.

Figure 2 shows the rotating vector for the specific Duffing system analysed. Also compared are the envelopes/displacements as well as the reciprocal of the time-varying period and instantaneous frequency $\mathrm{d}\psi/\mathrm{d}t$ (see Fig. 3). Both representations give very similar results. The exact relation between the phase plane representation, and the Hilbert-based one are discussed in detail in [8].

We now address the more common case of systems with damping. Let us assume that the system's equation can be described by the damping c and natural frequency ω_0 considered as functions of the envelope magnitude A. The phase plane representation is shown in Fig. 2(b) (dashed line).

It is possible to derive a related equation based on analytic signals. By adding to

$$\ddot{y} + c\dot{y} + \omega_0^2 y = 0 \tag{7}$$

an imaginary part equal to it's Hilbert transform, equation (7) can then be cast in the form

$$\ddot{Y} + c\dot{Y} + \omega_0^2 Y = 0. \tag{8}$$

Solving equation (8) enables us to compute $\omega_0(t)$ and $A(t)$ as a function of time [9, 10]. Eliminating the time variable from both functions results in $\omega_0^2(A)$, the classical backbone.

The function $A(t)$ for the case $\varepsilon = 5$ is shown in Fig. 1(b), together with the decaying oscillating response. The corresponding backbone is shown in Fig. 4.

We note that one effect of the non-linearity is the time variation of the system's natural frequency. While we have shown this specifically for the Duffing oscillator, time variable features are encountered for all non-linearities characterised by time-varying damping as well as time-varying natural frequency [9, 10]. It is important to emphasise that the system itself is a time invariant non-linear system, but that we are treating the impulse response function as a signal with time-variable characteristics.

4. SYSTEMS WITHOUT MEMORY

Systems whose input/output characteristics can be described by an algebraic expression in the time domain are memoryless: at any instant the response depends only on the current excitation, and not on its past. A general case would consist of a response y and excitation x related by $y = \mathscr{G}(x)$. For a sinusoidal excitation, the output will consist of a fundamental and harmonic components. Such a description is basically a frequency domain one, based on Fourier analysis. For this specific case another representation is possible, based on a variable instantaneous frequency signal. Let us assume that the signal y can be decomposed

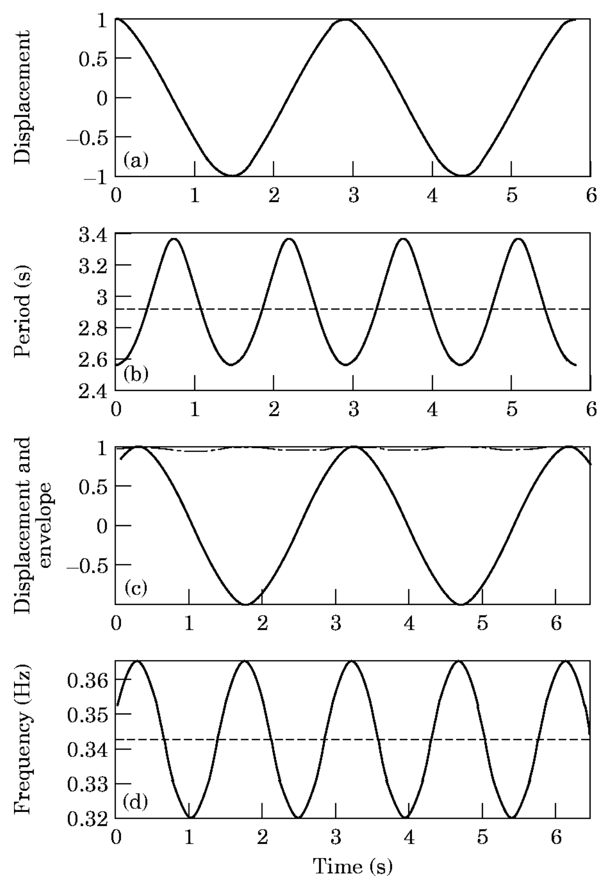

Figure 3. Representation of the Duffing equation solution ($\varepsilon = 5$, $h = 0$): (a) and (c) the solution; (b) the current period from the phase plane ($n = 100$); (d) the instantaneous frequency from the analytic signal.

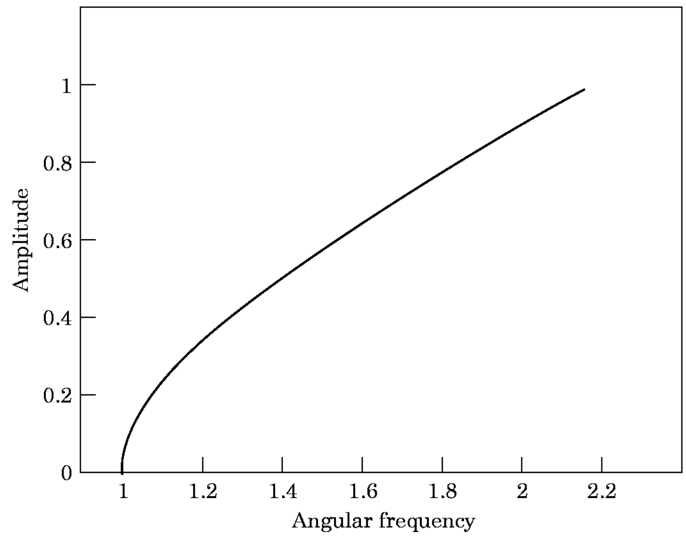

Figure 4. Backbone of the Duffing model ($\varepsilon = 5$).

into two components $y(t)\cos(\omega t) + k\cos(\omega t)$. Figure 5 shows a complex representation, based on rotating vectors, for $k < 1$.

$y(t)$ equals the horizontal projection of the resulting vector. In a polar representation, the vector is given by $A(t)$, $\psi(t)$ where both the envelope $A(t)$ and the instantaneous phase $\psi(t)$ are affected by the second component of the signal, of frequency 2ω. The instantaneous phase is given by

$$\psi(t) = \tan^{-1}\frac{\sin(\omega t) + k\sin(2\omega t)}{\cos(\omega t) + k\cos(2\omega t)} \tag{9}$$

We may actually use the standard definition of the instantaneous frequency. $\omega(t) = \mathrm{d}\psi/\mathrm{d}t$ for this signal, showing a variable frequency with time. This situation is depicted in Fig. 6.

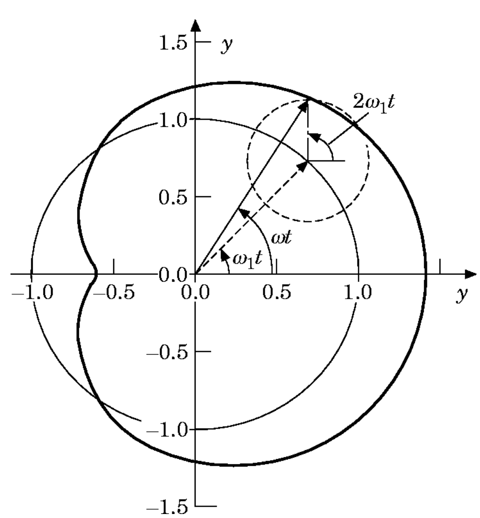

Figure 5. Analytic signal representation of a double component signal.

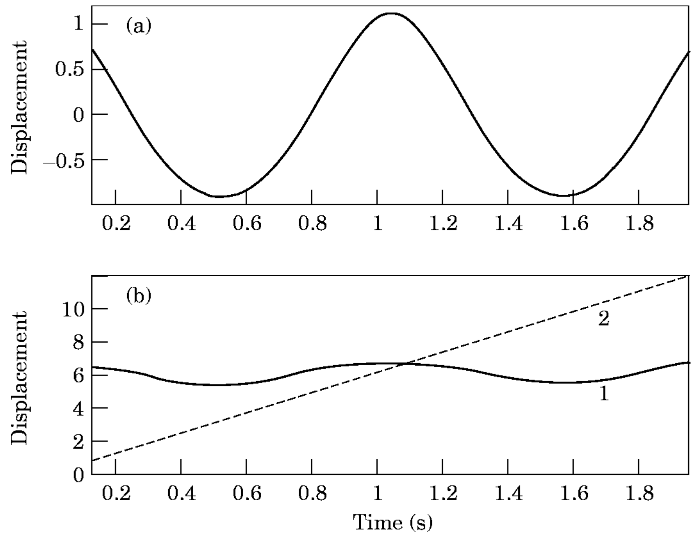

Figure 6. (a) Double component signal, (b) line 1, its instantaneous frequency; and line 2, instantaneous phase.

For the case where the signal function is not known explicitly, but given, say, by some measurement, the instantaneous frequency must be computed. In principle this should be possible again via the Hilbert transform. As an example let us investigate the cases

$$\begin{aligned} &\text{case 1:} \quad y = x; \\ &\text{case 2:} \quad y = x^3, \quad x = \cos(\omega t) \end{aligned} \tag{9}$$

Figures 7 and 8 show the time signal $y(t)$, the Hilbert transform, the instantaneous phase and frequency for both cases. It is evident that a completely different characterisation of the signal results. The instantaneous frequency is constant for case 1, and fluctuating for case 2. Some care is needed in the characterisation, as the end effects are due to the

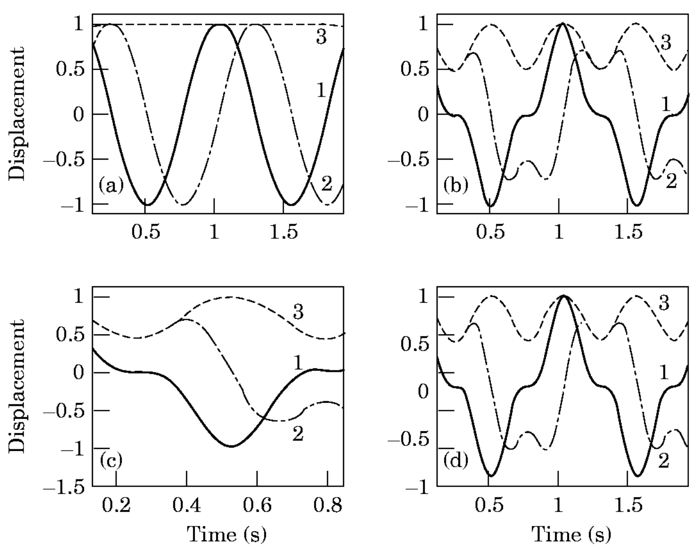

Figure 7. Signals and envelopes: (a) case 1, (b) case 2, (c) case 2 (short duration), (d) case 2 (noisy signal). Lines: 1, signal; 2, its Hilbert transform; 3, envelope.

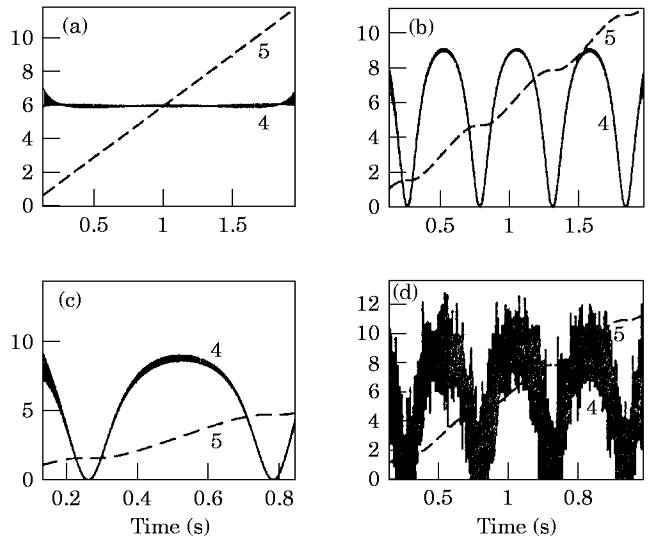

Figure 8. Instantaneous phase and frequency: (a) case 1, (b) case 2, (c) case 2 (short duration), (d) case 2 (noisy signal). Lines: 4, frequency; 5, phase.

difficulty of using the Hilbert transform around the time $t = 0$. We have chosen, for example, to ignore (discard) these end effects. The usefulness of our description has still to be determined; sensitivity to noise will certainly be of importance. One example of analysing our cases with some additive noise is shown in Fig. 8(d). The general behaviour seems unchanged. It would be interesting to compare this time domain analysis to a Fourier-based frequency domain one. Case 2 is analysed for two different signal durations. For the longer signal [see Figs 7 and 8(b)], the non-linearity is seen in the frequency domain

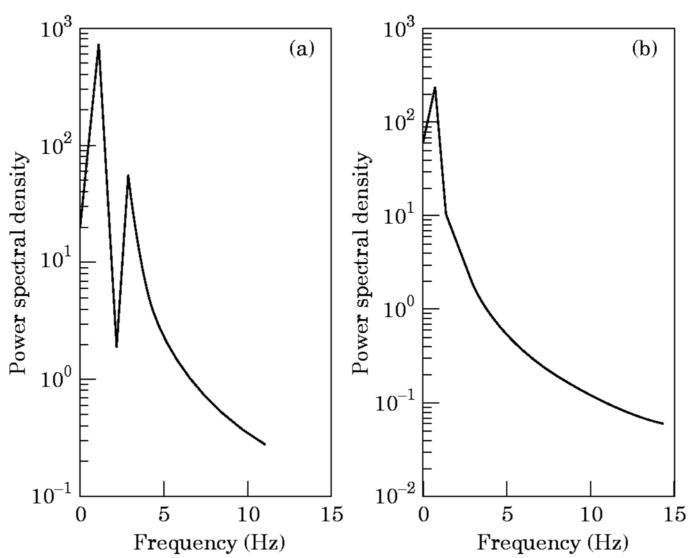

Figure 9. (a) Power spectrum of case 2 and (b) case 2 short duration.

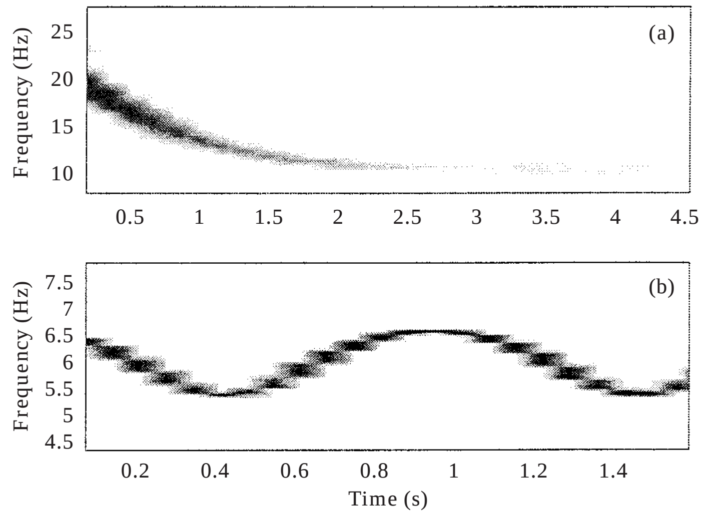

Figure 10. Time–frequency representation of (a) the Duffing solution and (b) double component signal.

as a third harmonic, and in the time domain by the fluctuating instantaneous frequency. For the short signal [see Figs. 7 and 8(c)], the resolution in the frequency domain is insufficient, but the non-linearity is still identifiable by the fluctuation in the instantaneous frequency. The limitation due to the uncertainty principle suggests the use of a different analysis domain, one of which could be the time–frequency characteristics.

5. CONCLUSION

In this short paper, we have addressed the possibility of characterising non-linear systems by the time–frequency variations of some system signals, i.e. signals representing the system's dynamics. Two examples were given. One, a Duffing oscillator showed time variations in the natural frequency of its impulse response. A more traditional representation of this behaviour is shown in Fig. 10(a). Here, time and frequency axes are used together with a grey scale for intensity variation, corresponding to the numerical example of Section 3. A second example consisted of the non-linear algebraic system (9). The system's characteristics were the response to a sinusoidal excitation. Figure 10(b) shows this, again in the more traditional time–frequency representation. The objective of signal processing is often to describe data from different perspectives (time domain, frequency domain, amplitude domain, etc.). Often we find advantages in specific representations. We have discussed the existence of such a representation, the time–frequency one, for signals representing systems, as compared to the traditional practice to apply time–frequency methods to system responses. For non-linear systems, specific patterns may be very indicative. It may be possible to reduce specific error mechanisms. For example, in Section 4 the effect of the Uncertainty Principle related resolution (in frequency analysis) seemed to be reduced in our analysis. The Hilbert transform seems to have much potential for this approach. Our analysis is of course of a preliminary nature, and many issues have to be investigated rigorously.

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