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NON-LINEAR FREE VIBRATION IDENTIFICATION VIA THE HILBERT TRANSFORM

M. Feldman

Faculty of Mechanical Engineering, Technion—Israel Institute of Technology, Haifa, 32000, Israel

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A time domain non-parametric method for non-linear vibration system identification based on the Hilbert transform is introduced. The method is demonstrated using computer simulations of different types of non-linear elastic and damping dynamic systems.

1. INTRODUCTION

The Hilbert Transform (HT) and its properties, applied to the analysis of linear and non-linear vibrations, are discussed at great length in [1]. The application of the HT to a signal provides some additional information about amplitude, instantaneous phase and vibration frequency. This information is valid when applied to the analysis of non-linear vibration motions [2]. Moreover, it has been noted that the HT should also be used for solving an inverse problem—the problem of vibration system identification.

Previous results of applying the HT to time domains for non-linear vibration system identification are presented in reference [3]. The simplest natural vibration system, having mass and a linear stiffness element, in a time domain gives rise to pure harmonic motion. In the presence of non-linear elastic forces, the natural frequency will depend decisively on the amplitude of vibration. Real free vibration always gradually decreases in amplitude due to system energy losses. Energy dissipation lowers the instantaneous amplitude according to a non-linear dissipative function. As non-linear dissipative and elastic forces have distinct effects on free vibrations, the HT identification methodology [4] enables the determination of some aspects of the behavior of these forces. For this identification in the time domain it was proposed that relationships be constructed between the damping coefficient (or decrement) as a function of amplitude plus relationships between the instantaneous frequency and amplitude (system backbone) [5].

Note, however, that previous work incorporated a theoretical assumption about a given vibration signal, which should have a slowly varied envelope (weak non-linearity condition). This paper describes a generalized HT identification approach for non-linear free vibration of one or two degrees of freedom system without the assumption of weak non-linearity.

2. APPLICATION OF THE HT

2.1. MAIN PROPERTIES OF THE HT

The single-value extraction (demodulation) of an envelope and other instantaneous functions of a signal is based on the *Hilbert integral transform* [1]. The HT of a

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real-valued function $x(t)$ extending from $-\infty$ to $+\infty$ is a real-valued function defined by

$$\mathbf{H}[y(t)] = \tilde{y}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y(\tau)}{t-\tau}\, \mathrm{d}\tau,$$

where $\tilde{y}(t)$ is the HT of the initial process $y(t)$, and the meaning of the integral implies its Cauchy principal value. Thus $\tilde{y}(t)$ is the convolution integral of $y(t)$ with $(1/\pi t)$, written as $\tilde{y}(t) = y(t) * (1/\pi t)$. The double HT yields the original function with an opposite sign, and hence it carries out shifting of the initial signal in $-\pi$. The power (or energy) of a signal and its HT are equal. For $n(t)$ low-pass and $y(t)$ high-pass signals with non-overlapping spectra [1],

$$\mathbf{H}[n(t)y(t)] = n(t)\tilde{y}(t). \tag{1}$$

More generally, the HT of a multiplication of two varying functions with overlapping spectra can be written in the form of a sum of two parts [6]

$$\mathbf{H}[n(t)y(t)] = \mathbf{H}\{[\bar{n}(t) + n_1(t)]y(t)\} = \bar{n}(t)\tilde{y}(t) + \tilde{n}_1(t)y(t). \tag{2}$$

where $\bar{n}(t)$ is the slow (low-pass), $n_1(t)$ is the fast (high-pass) signal component, and $\tilde{n}_1(t)$ is the HT of the fast component. The proof of the decomposition of a signal into a sum of low- and high-pass terms, based on Bedrosian's theorem the HT of a product, can be found in reference [7]. For example, the HT of the square of the harmonics $y = A\cos\phi$ is equal to $\mathbf{H}[y^2(t)] = \mathbf{H}[y(t)y(t)] = \tilde{y}(t)y(t) = A^2\sin(2\phi)/2$, and the HT of the cube of the harmonics is equal to $\mathbf{H}[y^3(t)] = \mathbf{H}[y^2(t)y(t)] = A^3(3\sin\phi + \sin 3\phi)/4$.

2.2. ANALYTICAL SIGNAL REPRESENTATION

The non-linear restoring force of a second order conservative system can be represented as the multiplication of a varying non-linear natural frequency $\omega_0(y)$ and the system solution

$$\ddot{y} + k(y) = \ddot{y} + \omega_{01}^2(y)y = 0. \tag{3}$$

Assume, according to equation (2), that the varying non-linear natural frequency can be separated into two different parts. The first part $\bar{\omega}_0$ is much slower and the second component $\omega_1(y)$ is faster than the system solution, so the equation of motion will be

$$\ddot{y} + [\overline{\omega_0^2} + \omega_1^2(y)]y = 0. \tag{4}$$

Now, according to the multiplication property of the HT, equation (2) we use the HT for both sides of equation (4) $\tilde{\ddot{y}} + \overline{\omega_0^2}\,\tilde{y} + \widetilde{\omega_1^2}(y)y = 0$. Multiplying each side of the last equation by j and adding it to the corresponding sides of equation (4) we obtain a differential equation with analytical signal form $\ddot{Y} + \overline{\omega_0^2}\,Y + [\omega_1^2 + \mathrm{j}\widetilde{\omega_1^2}]y = 0$, where $Y = y + \mathrm{j}\tilde{y}$. This complex equation can be transformed to the commonly accepted form

$$\ddot{Y} + \mathrm{j}\delta_0 Y + \omega_0^2 Y = 0, \qquad \omega_0^2 = \overline{\omega_0^2} + \frac{\omega_1^2 y^2 + \widetilde{\omega_1^2}\, y\tilde{y}}{A^2}, \qquad \delta_0 = \frac{\widetilde{\omega_1^2}\, y^2 - \omega_1^2\, y\tilde{y}}{A^2}, \tag{5–7}$$

where ω_0^2 is the varying natural frequency function, and δ_0 is the fast varying fictitious damping parameter. The equation obtained for the non-linear system has a varying natural frequency consisting of a slow component, $\overline{\omega_0}$ and a fast component, as well as a fast varying fictitious damping parameter δ_0. It should be noted that this non-stationary equation is not a real equation of motion, but rather an artificial equation which produces the same non-linear vibration signal.

Let us consider a general case of SDOF conservative systems having the non-linear elastic characteristics that could be expressed as a power series

$$k(y) = (\varepsilon_1 + \varepsilon_3 y^2 + \varepsilon_5 y^4 + \cdots)y = y \sum_{1}^{n} \varepsilon_{2l-1} y^{\pm(2l-2)} \qquad (l = 1, 2, 3 \ldots n). \tag{8}$$

In particular, a non-linear system described by Duffing's equation has only the linear and the positive hard cubic spring $k(y) = \omega_0^2 (1 + \varepsilon_3 y^2) y$. After substituting the expanded elastic characteristics equation (8) together with the solution $y = A \cos \psi$ into the varying natural frequency equation (6), one can derive the particular type of the natural frequency function. For example, in the presence of a cubic non-linearity, the speed of the instantaneous frequency oscillation is twice that of the main vibration frequency (Figure 1(a)). Generally, after averaging the natural frequency function, we obtain an expression for the slow component of the natural frequency:

$$\langle \omega_0^2 \rangle = T^{-1} \int_0^T \omega_0^2(t)\, \mathrm{d}t = \varepsilon_1 + \tfrac{3}{4} \varepsilon_3 A^2 + \tfrac{5}{8} \varepsilon_5 A^4 + \cdots = 2^{-2l+2} \binom{2l-1}{l-1} A^{2l-2}. \tag{9}$$

It is important that the expression obtained in equation (9), corresponding to numeric coefficients, repeats the structure of the initial non-linear elastic characteristics in equation (8). In other words, the estimated average natural frequency function $\langle \omega_0^2 \rangle$ includes all information about the initial system and can be used for system identification.

It is also important that an average value of the fast varying damping parameter equation (7) is equal to zero. Consequently, the fast varying fictitious damping force does not affect the real average damping force. The obtained results explain the fact that the instantaneous frequency as well as the backbones of non-linear systems after the HT analysis exhibits an unusually fast oscillation (modulation) form [2].

By analogy, one can write the corresponding expressions for initial non-linear damping in the system. An average damping function is also obtainable after the HT repeats the

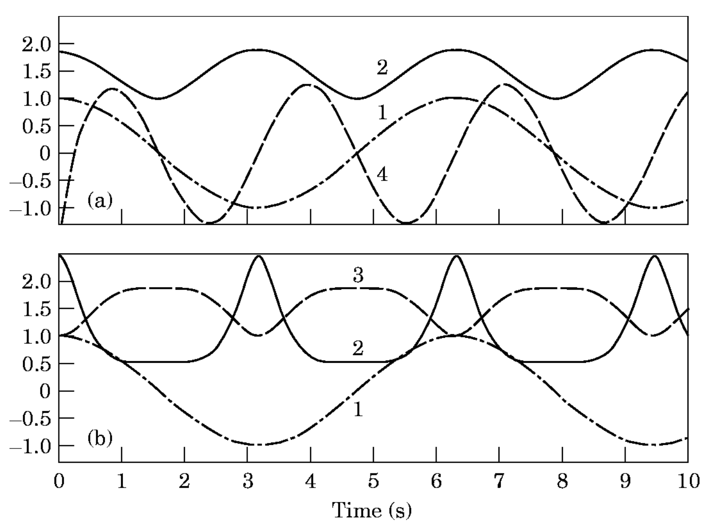

Figure 1. The Duffing equation ($\varepsilon = 5$) representation: (a) analytical signal; (b) phase plane. 1, Solution; 2, frequency; 3, radius vector; 4, damping.

structure of the initial non-linear damping force characteristics and can be used for system identification.

2.3. PHASE PLANE REPRESENTATION

Resultant fast oscillation of the natural frequency does not associate with the HT representation. It is an interesting and an essential effect of free vibration of non-linear systems. The natural frequency oscillation also takes place according to the classic phase approach. Traditionally, a new variable $\dot{y}$ is introduced, enabling the exclusion of time from the equation of motion although y and $\dot{y}$ are still time dependent, so

$$\ddot{y} = \frac{\mathrm{d}\dot{y}}{\mathrm{d}t} = \frac{\mathrm{d}\dot{y}}{\mathrm{d}y}\dot{y}.$$

In the new co-ordinates, equation (3) takes the following form:

$$\frac{\mathrm{d}\dot{y}}{\mathrm{d}y} = \frac{k(y)}{\dot{y}}.$$

Using the new variable $\dot{y}$ is a traditional way of studying the motion of an oscillator by representing this motion on the y–$\dot{y}$ plane, where y and $\dot{y}$ are orthogonal Cartesian co-ordinates. After variable separation and integration, the phase plane takes the form $\int \dot{y}\,\mathrm{d}\dot{y} = \int k(y)\,\mathrm{d}y$. The phase plane radius vector $r(\phi)$ for the non-linear system modulates from its minimum value to its maximum value

$$r = \sqrt{y^2 + \dot{y}^2}. \tag{10}$$

In addition, the phase plane angular frequency fluctuates between a maximum and a minimum as the radius vector rotates between the y- and $\dot{y}$-axes of the phase plane

$$\omega_{p0} = d\left[\arctan\frac{\dot{y}(\phi)}{y(\phi)}\right]\mathrm{d}\phi. \tag{11}$$

To illustrate this interesting phenomenon of both the radius and the frequency of phase plane modulation, let us consider an example of the Duffing equation:

$$\ddot{y} + (1 + \varepsilon y^2)y = 0, \tag{12}$$

where ε is the non-linear parameter. The radius vector square of the Duffing phase plane derived from equation (10) appears as $r^2(t) = -0{\cdot}5\varepsilon y^4 + r_0^2 + 0{\cdot}5\varepsilon r_0^4$, where $r_0 = \max(y)$. The obtained modulated radius vector together with its frequency and the solution in the form $y = r\cos\phi$ (according to equations (10) and (11) are shown in Figure 1(b)). It is clear that the radius vector and the frequency oscillate twice as fast as those of non-linear solutions. In the general case of non-linear systems, the radius vector and the frequency are fast varying functions of a phase angle.

2.4. TWO-COMPONENT SIGNAL REPRESENTATION

Consider a non-linear solution that consists of a composition of two quasi-harmonics, each with a slow variable amplitude and frequency in the time domain. In this case, the signal can be modelled as a weighted sum of monocomponent signals, each with its own instantaneous frequency and amplitude function: that is,

$$Y(t) = A_1\,\mathrm{e}^{\mathrm{j}\int_0^t \omega_1\,\mathrm{d}t} + A_2\,\mathrm{e}^{\mathrm{j}\int_0^t \omega_2\,\mathrm{d}t}, \tag{13}$$

with A_1, A_2, ω_1 and ω_2 being unknown functions in the time domain. One of the questions arising immediately from this representation is: How will the combined signal $Y(t)$ be separated into its two initial parts? The HT also plays an important role in the signal decomposition and leads to practical results [8, 9].

The envelope and the instantaneous frequency of the double-component vibration signal $Y(t)$ are

$$A(t) = \left[A_1^2 + A_2^2 + 2A_1 A_2 \cos\left(\int (\omega_2 - \omega_1)\, \mathrm{d}t \right) \right]^{1/2}, \tag{14}$$

$$\omega(t) = \omega_1 + \frac{\left[A_2^2 + A_1 A_2 \cos\left(\int (\omega_2 - \omega_1)\, \mathrm{d}t \right) \right]}{(\omega_2 - \omega_1)^{-1} A^2}. \tag{15}$$

From equation (14), it can be seen that the signal envelope consists of two different parts; that is, a slowly varying part including the sum of the component amplitudes squared, $A_1^2(t)$ and $A_2^2(t)$, and a rapidly varying (oscillating) part, the multiplication of these amplitudes with function cos of the relative phase angle between two components. Eliminating the oscillating part $\cos[\int (\omega_2(t) - \omega_1(t))\, \mathrm{d}t]$ from equations (14) and (15), we shall find the equation between the signal instantaneous characteristics (envelope and frequency) and the initial four parameters of the signal components:

$$A^2(t) = \frac{(A_1^2 - A_2^2)(\omega_2 - \omega_1)}{\omega_1 + \omega_2 - 2\omega}, \qquad A_1 \neq A_2, \quad \omega_1 \neq \omega_2. \tag{16}$$

Equation (16) determines the signal envelope A as a function of instantaneous frequency ω in the form of an hyperbola, the length and curvature of which depends on four initial parameters.

2.4.1. *Estimating the amplitude of each component*

Assume that the relative phase angle $\int (\omega_2(t) - \omega_1(t))\, \mathrm{d}t$ is large enough to show a number of "beatings" of two quasi-harmonics:

$$\left| \int_0^t (\omega_2(t) - \omega_1(t))\, \mathrm{d}t \right| \gg 2\pi. \tag{17}$$

For such a case, it is possible to separate the slow and the fast (oscillating) parts of a signal envelope (mentioned above) by using an ordinary filtration in the frequency domain. Thus only the fast part $A_f^2(t)$ will be retained after high pass filtration of the square of the signal envelope (see equation (14)):

$$A_f^2(t) = 2A_1(t) A_2(t) \cos\left(\int_0^t (\omega_2(t) - \omega_1(t))\, \mathrm{d}t \right). \tag{18}$$

This new function is now just a monocomponent signal. After repeating application of the Hilbert transform [8], the new envelope extraction is readily achieved $A_m(t) = 2A_1(t)A_2(t)$ and the new instantaneous frequency is then

$$\pm\omega_m(t) = \omega_2(t) - \omega_1(t), \tag{19}$$

where a positive sign of $\omega_m(t)$ should be used in assuming that the first component frequency is lower than the latter ($\omega_1(t) < \omega_2(t)$), or a minus sign should be used in the opposite case.

Then, using an algebraic transform, we obtain a simple formula for amplitude components estimation:

$$A_1(t) = 0{\cdot}5[(A_s^2 + A_m^2)^{1/2} \pm (A_s^2 - A_m^2)^{1/2}],$$
$$A_2(t) = 0{\cdot}5[(A_s^2 + A_m^2)^{1/2} \mp (A_s^2 - A_m^2)^{1/2}], \tag{20}$$

where $A_s^2(t) = A^2(t) - A_f^2(t) = A_1^2(t) + A_2^2(t)$ is a slow part of the square of the signal envelope. An upper sign before the square root should be used in assuming that the first component is stronger than the other one ($A_1(t) > A_2(t)$), or a lower sign should be used in the opposite case.

2.4.2. *Instantaneous frequency estimating*

The instantaneous frequency of each component can also be estimated after repeated application of the Hilbert transform from equations (16) and (19):

$$\omega_1(t) = \omega(t) \pm \omega_m(t)\left(\frac{A_1^2(t) - A_2^2(t)}{2A(t)} - 1\right), \qquad \omega_2(t) = \omega_1(t) \pm \omega_m(t), \tag{21}$$

where $A_1(t)$, $A_2(t)$, $\omega_1(t)$ and $\omega_2(t)$ are the initial parameters of the quasi-harmonics, and $\pm\omega_m(t)$ is the instantaneous frequency of the oscillating part of the signal envelope (see equation (19)).

Equation (16) determines the signal envelope A as a function of instantaneous frequency ω in the form of a hyperbola, the length and curvature of which depends on the four initial parameters of the biharmonics. If, in reality, the initial amplitude components decrease in time, the estimated HT backbone (without averaging) stretches to a spiral function instead of a short hyperbola.

The use of the developed technique could result in a more precise estimation of both the amplitude and the frequency of each harmonic component of double component vibration signals. In the case of two-degree-of-freedom non-linear systems, the technique realizes time domain decomposition, which is able to estimate each mode of the non-linear vibration system.

3. FEATURES OF THE HT IDENTIFICATION

3.1. NON-LINEAR BACKBONE AND DAMPING CURVE IDENTIFICATION

A vibration signal, suitable for the HT identification, should be a monocomponent signal derived from a SDOF system directly, or obtained from a multi-DOF system after a special decomposition or after band-pass filtration. This initial signal is

$$y(t) = A(t)\cos\psi(t), \tag{22}$$

where $y(t)$ is the vibration signal (a real valued function), $A(t)$ is an envelope (the instantaneous amplitude) and $\psi(t)$ is an instantaneous phase, assumed to be a free solution of a non-linear vibration system. Taking into account the analytical signal representations based upon equation (5) enables one to consider this equivalent equation of motion, so as to estimate the instantaneous natural frequency and the instantaneous damping

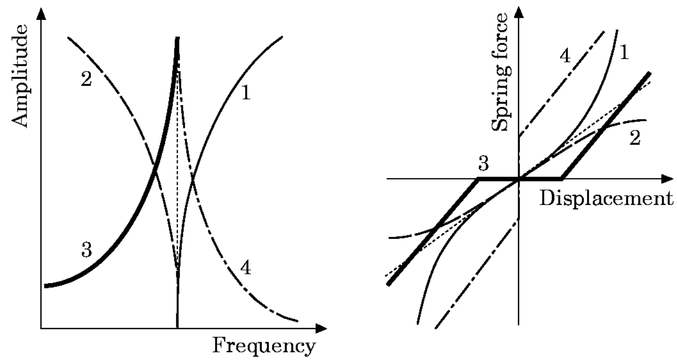

Figure 2. Typical non-linear backbones (a) and spring force characteristics (b). 1, Hard; 2, soft; 3, backlash; 4, pre-loaded spring.

coefficient. A general form of an initial differential equation of motion in the analytical signal form could be written for frequency dependent (viscous) damping [4] as

$$\ddot{Y} + 2h_0(A)\dot{Y} + \omega_0^2(A)Y = 0, \tag{23}$$

and, for frequency independent (structural) damping, as

$$\ddot{Y} + \omega_0^2(A)\left[1 + \mathrm{j}\,\frac{\delta(A)}{\pi}\right]Y = 0, \tag{24}$$

where $Y = y(t) + \mathrm{j}\tilde{y}(t)$ is the system solution in the analytical signal form, h_0 is the instantaneous damping coefficient, ω_0 is the instantaneous undamped natural frequency, $\delta = 2\pi\zeta$ is the logarithmic decrement and ζ is the damping ratio. This equation of motion will have rapidly varying coefficients that satisfy the envelope and instantaneous frequency of the non-linear oscillating solution (equations (5)–(7)). Solving two new equations separately for the real and imaginary parts of equations (23) or (24), we can write the expression for the rapidly varying coefficients as a functions of a first and a second derivative of the signal envelope and the instantaneous frequency:

$$\omega_0^2(t) = \dot{\psi}^2 - \frac{\ddot{A}}{A} + \frac{2\dot{A}^2}{A^2} + \frac{\dot{A}\ddot{\psi}}{A\dot{\psi}},$$

$$h(t) = -\frac{\dot{A}}{A} - \frac{\ddot{\psi}}{2\dot{\psi}}, \qquad \delta(t) = -\frac{2\pi\dot{A}\dot{\psi}}{A\omega_0^2} - \frac{\pi\ddot{\psi}}{\omega_0^2}, \tag{25}$$

where $\omega_0(t)$ is the instantaneous undamped natural frequency of the system, $h(t)$ is the instantaneous damping coefficient of the system, $\delta(t)$ is the instantaneous logarithmic decrement, and $\dot{\psi}$ and A are the instantaneous frequency and envelope (amplitude) of the vibration with their first and second derivatives. Algebraically, equation (25) means that the HT identification method FREEVIB uses initital displacement, velocity and also acceleration at a time [5]. A traditional theoretical backbone of a non-linear system is a dependency between the average free vibration frequency $\langle\omega_0\rangle$ (corresponding to the average total cycle of vibration) and the displacement amplitude. Some typical non-linear examples of backbone representation will be considered further in the paper (see Figure 2).

The proposed direct time domain method based on the HT allows a direct extraction of the linear and non-linear parameters of the system from the measured time signal of output. The resulting non-linear algebraic equations (25) are rather simple and do not depend on the type of non-linearity that exists in the structure. When applying this direct method for transient vibration, the instantaneous modal parameters are estimated directly. That enables us to consider an inverse identification problem; namely, the problem of estimation of the initial non-linear elastic and damping force characteristics.

3.2. NON-LINEAR FREQUENCY RESPONSE FUNCTION CONSTRUCTION

As a result of the FREEVIB method, the set of duplet modal parameters (instantaneous natural frequency $\omega_0(A)$ and instantaneous damping $h(A)$) of each natural mode of vibration is defined. The obtained modal model leads to a description of the structure's behavior as a set of vibration modes. This model could be defined as a set of natural frequencies with corresponding unit mass and modal damping factors. However, it is convenient to present an analysis of the structure's response in a standard form and to describe this with the Frequency Response Function (FRF). The standard excitation is that of sinusoidal force applied to the input of the system at every frequency in the specified range. Thus the tested system frequency response function can be written as a SDOF frequency response function:

$$A = [2A_{max}\, h(A)] \Bigg/ \left\{ \omega_0(A) \sqrt{\left[1 - \frac{\omega^2}{\omega_0^2(A)}\right]^2 + \frac{4h^2(A)\omega^2}{\omega_0^2(A)}} \right\} \tag{26}$$

where A is the steady state vibration amplitude (proportional to the magnitude of the FRF), $\omega = 2\pi f$ is the angular frequency of vibration, $\omega_0(A) = 2\pi f_0(A)$ is the angular natural undamped frequency as a function of amplitude, and $h(A)$ is the damping coefficient, as a function of amplitude. In order to plot the FRF of the tested system after the FREEVIB method, equation (26) should be further inverted:

$$\omega^2 = \omega_0^2(A) - 2h^2(A) \pm 2\omega_0(A)h(A)\sqrt{\frac{A_{max}^2}{A^2} - 1 + \frac{h^2(A)}{\omega_0^2(A)}},$$

$$0 \leqslant A \leqslant A_{max}\left[1 - \frac{h^2(A)}{\omega_0^2(A)}\right]^{-1/2}. \tag{27}$$

Using the last equation we can plot the frequency response function as separate curves together with the system backbone (skeleton) curve.

3.3. FORCE CHARACTERISTIC INTERPRETATION

3.3.1. *Decomposition technique*

Consider the case of a conservative system with an initial non-linear spring. According to equation (23) the real non-linear elastic force will produce two different fast varying fictitious members (elastic and damping). The real restoring force includes both the fast hysteretic damping and the fast elastic force. Thus, the initial non-linear spring force $\omega_0(y)^2 y$ is split into two terms and, by summing over the terms, the initial non-linear force characteristics can be extended. Therefore a simple composition of these two components of the motion equation (equation (23)) will result in the real non-linear force characteristics:

$$k[y(t)] = 2h_0(t)\dot{y} + \omega_0^2(t)y, \tag{28}$$

where $k[(y)]$ is the realelastic instantaneous force, $h_0(t)$ is the instantaneous damping coefficient and $\omega_0(t)$ is the instantaneous undamped natural frequency.

3.3.2. *Scaling technique*

The obtained expression for the average value of the natural frequency function qualitatively repeats the structure of the initial non-linear elastic characteristics. Due to the estimation of the polynomial coefficients of the average natural frequency by equation (9), we can simply reconstruct the initial non-linear elastic characteristics in equation (8). An additional scaling technique could be considered based upon the total energy, which is constant for conservative vibration systems. During free vibration, the energy of the system at each moment is partly kinetic, partly potential and partly fictitious alternating positive or negative damping. There are moments in time at which all of the energy is stored mainly as a strain energy of elastic deformation and the fictitious damping energy is equal to zero. These points correspond to the maximum of the elastic force. Using, for instance, the biharmonics representation of non-linear vibrations (equation (13)) one can show that these time points correspond to the maximum displacement. This illustrates an important property of a conservative vibration system, that around every peak point of the displacement the corresponding value of the velocity is equal to zero, and *vice versa*. Therefore around every peak point of the displacement, the contribution of the velocity in the varying instantaneous elastic force is negligibly small: $y(t_i) = A(t_i)$, $\dot{y}(t_i) = 0$. The average value of the envelope of the fictitious elastic force $\langle\omega_0^2(A)\rangle A$ will have a small bias relative to the maximums of the spring force. The number of the peak points is far less than the total number of points of a vibration signal, but it enables us to calculate a ratio set of average and maximum (real) values of the elastic force. The obtained set is a scale function that can adjust the average force characteristics. Finally, we normalize the elastic as well as the damping forces as functions of the vibration envelope.

A complete understanding of the HT identification method entails noting its current limitations: (i) each obtained force characteristics is a relative characteristic, dealing only with a unit mass of the vibration system; (ii) each obtained force characteristic is taken to be a static symmetric characteristics; and (iii) a tested system is certain to be a light damped system with an underdamped damped term to produce an oscillating motion.

In general, SDOF systems can include several elastic and damping individual elements, combined integrally through parallel and/or series connections. If each individual element of an SDOF system is known, it is usually possible to determine the resultant or equivalent system force characteristics. However, the inverse problem has no unique solution. Let us take an example of a system with two spring elements connected in parallel, the first of which is a backlash and the second just a linear spring. The corresponding equivalent force characteristics will have a bi-linear form with a linear section for displacement, less than clearance and with different linear sections for a higher amplitude. If only information on equivalent force characteristics is available, it is not possible to reconstruct the initial system. In our example it can be a bi-linear element, or two different initial elements.

The HT method, as a non-parametric method, forms resultant non-linear elastic and damping force characteristics by direct extracting vibration system backbones and damping curves. The non-linear spring force function, identified from vibration motion of the SDOF system, will completely correspond to the initial system static elastic force characteristic per unit mass. In the case of system model identification, one should use additional information regarding the model structure and its element combination.

4. NON-LINEAR VIBRATION IDENTIFICATION

4.1. FREE VIBRATION LARGE AMPLITUDE NON-LINEAR BEHAVIOR

Most well known cases of non-linearity occur in large amplitude oscillations of elastic systems; for instance, non-linear spring elements with hardening or softening restoring force, or non-linear damping quadratic or cubic force. Whereas the amplitudes of vibration are large, the occurrence of these spring or damping non-linearities cannot be ignored.

As an example, we refer to the non-linear system with a soft spring and a cubic damping characteristic:

$$\ddot{y} + c_1 \dot{y} + c_2 \dot{y}^3 + k_1 y - k_2 y^3 = 0,$$

where $k_1 = 1$, $k_2 = 0{\cdot}9$, $c_1 = 0{\cdot}05$, $c_2 = 0{\cdot}3$ and $y(0) = 1$. A simulation of a free vibration signal was carried out by using a non-zero initial displacement as shown in Figure 3(a). Both the obtained backbone and the damping curve (Figures 3(b) and (c)) have typical non-linear form. Spring and damping force characteristics, estimated after the HT identification, practically coincide with the theoretical characteristics. In Figure 4(a) is shown an example of the instantaneous elastic force identification for the soft spring system. Figure 4 includes the results of the identification according to equation (28) together with the initial cubic force characteristics $k(y) = (k_1 - k_2 y^2)y$, but these lines agree so closely that there is virtually no difference.

4.2. FREE VIBRATION SMALL AMPLITUDE NON-LINEAR BEHAVIOR

There are cases in which vibration systems show their specific non-linear behavior only in a small amplitude range of vibrations. A dynamic system with backlash is a typical example of such mechanical non-linear systems, because for large amplitude values it operates like a linear system with a constant natural frequency. Only for small vibration amplitudes commensurable to a clearance value will the system display its non-linear properties, where natural frequency decreases as amplitude decreases.

A mechanical system with a pre-loaded (pre-compressed) restoring force is another example of non-linearity in the small amplitude range. Actually, for large vibration

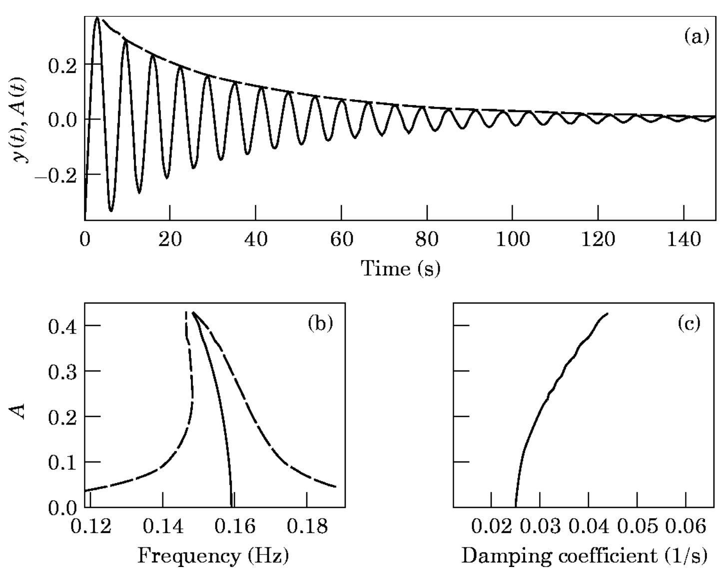

Figure 3. High amplitude free vibration (a), backbone and FRF (b) and the damping curve (c).

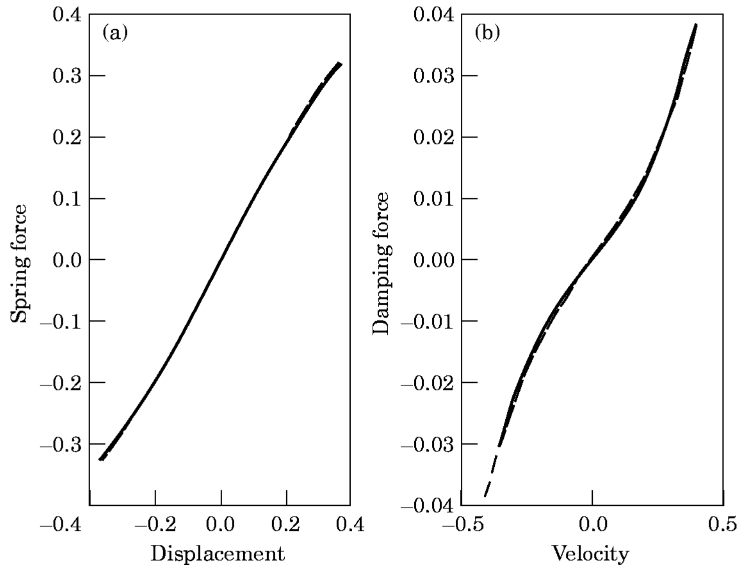

Figure 4. The high amplitude estimated spring force (a) and damping force (b) characteristics.

amplitudes natural frequency practically does not depend on vibration amplitude. Only for small amplitudes of vibration motion commensurable to a pre-compressed deformation will the natural frequency increase notably.

Among vibration systems with non-linear damping characteristics, there are structures which show non-linearities only in a small amplitude range. As an example, we mention a system with Coulomb or dry damping, the plot of the logarithmic decrement of which versus the vibration amplitude is a monotonic hyperbola [5]. The presence of dry together with viscous damping means that only for small vibration amplitudes does the logarithmic decrement increase extremely.

By way of illustration, consider a non-linear system with backlash and dry damping:

$$\ddot{y} + c_1 \dot{y} + c_2 \operatorname{sgn}(\dot{y}) + k(y)y = 0,$$

$$k(y) = \begin{cases} k[1 - (\Delta/y)\operatorname{sgn}(y - \Delta)] & \text{if } |y| > \Delta, \\ 0 & \text{if } |y| \leqslant \Delta, \end{cases}$$

where $c_1 = 0{\cdot}017$, $c_2 = 0{\cdot}007$, $k = 1$, $2\Delta = 0{\cdot}3$ and $y(0) = 8$.

Simulation data (displacement) together with its envelope is shown in Figure 5(a), and the obtained backbone and damping curve in Figures 5(b) and (c). The comparison of simulated and estimated force characteristics is given in Figure 6. The obtained non-linear backbone and elastic force characteristics practically coincide with the corresponded theoretical system characteristics. Using the proposed HT analysis in the time domain, we can extract both the instantaneous undamped frequency and also the real non-linear elastic force characteristics.

4.3. IDENTIFICATION OF TWO-DEGREE-OF-FREEDOM SYSTEM

Generally, systems composed of several masses, non-linear springs and dampers require more complicated representation. Consider the equations of motion, for example, for a coupled 2-DOF system without damping:

$$m_{11}\ddot{y}_1 + k_{11}(y)y_1 + k_{12}(y)y_2 = 0, \qquad m_{22}\ddot{y}_2 + k_{21}(y)y_1 + k_{22}(y)y_2 = 0, \tag{29}$$

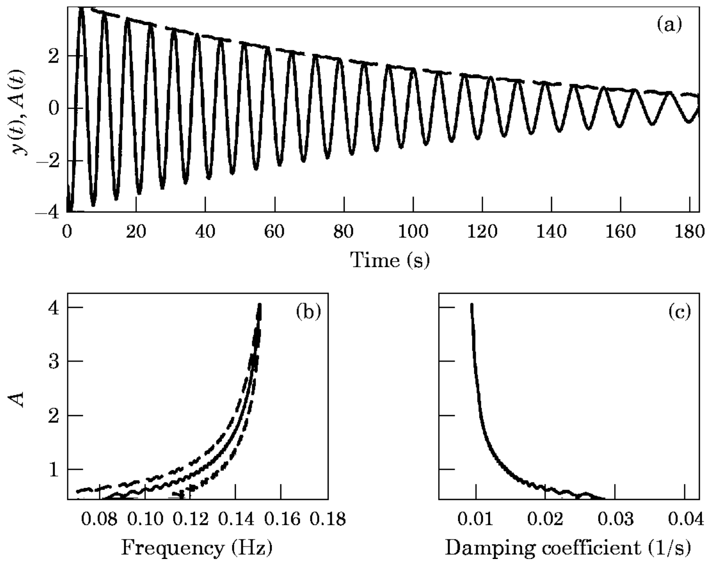

Figure 5. Low amplitude free vibration (a), backbone and FRF (b) and the damping curve (c).

where $k_{11}(y)$, $k_{22}(y)$, $k_{12}(y)$ and $k_{21}(y)$ are non-linear springs or coupling spring functions. Assume that non-linear spring elements are relatively small and that the system behaves like a quasi-linear system. In this case we can present each weakly non-linear subsystem in the neighborhood of some amplitude, as an approximate subsystem with equivalent linearized elements: $k_i(y) \approx k_i^*$:

$$m_{11}\ddot{y}_1 + k_{11}^* y_1 + k_{12}^* y_2 = 0, \qquad m_{22}\ddot{y}_2 + k_{21}^* y_1 + k_{22}^* y_2 = 0. \tag{30}$$

After using any decoupling technique, we will have several corresponding decoupled equations of motion. For linear systems, each obtained natural frequency is constant and

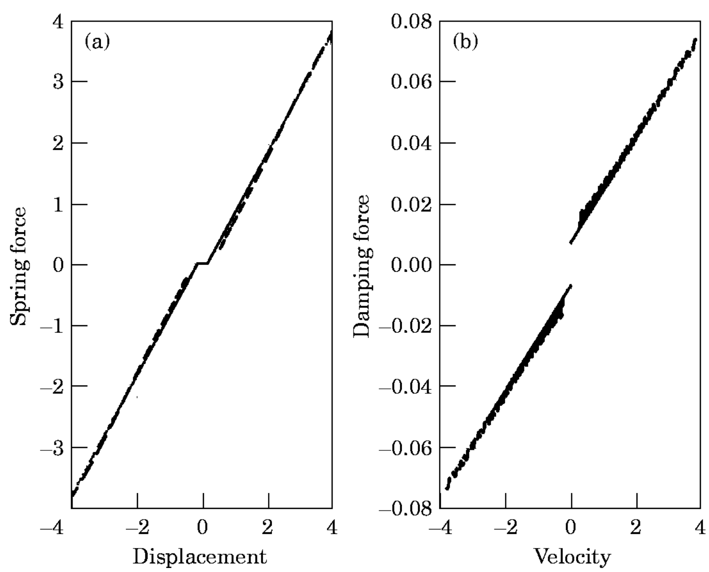

Figure 6. The low amplitude estimated spring force (a) and damping force (b) characteristics.

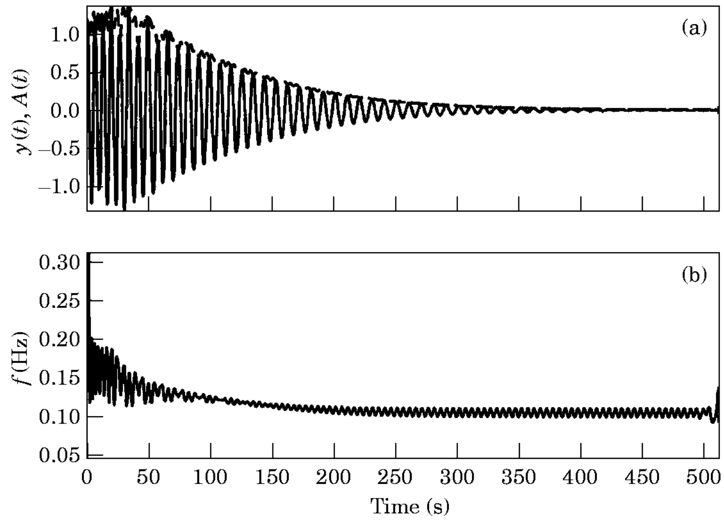

Figure 7. Double component free vibration (a) and the instantaneous frequency (b).

differs from the partial subsystem natural frequency according to the decoupling co-ordinate transformation. In the case of non-linear systems, natural frequencies become functions of amplitude as well, since each natural frequency, corresponding to the normal co-ordinate, will be controlled by the linearized elements (equation (31)).

Making this decoupling technique for a set of amplitudes and for the corresponding equivalent springs, we can extract the non-linear system backbone together with the corresponding force characteristics for each quasi-linear mode. According to the decoupling co-ordinate transformation, the obtained non-linear force characteristics differ from their initial subsystem form, but do hold their main qualitative representation.

An example of the non-linear system considered here is a 2-DOF hard spring system with equation of motion,

$$\ddot{y}_1 + c_1 \dot{y}_1 + k_1 y_1 + k_3 y_1^3 + k_2 (y_1 - y_2) = 0,$$

$$\ddot{y}_2 + c_2 \dot{y}_2 + k_2 y_2 - k_2 (y_1 - y_2) = 0,$$

where $k_1 = 1$, $k_2 = 1$, $k_3 = 7{\cdot}0$, $c_1 = 0{\cdot}1$, $c_2 = 0$, $y_1(0) = 1{\cdot}0$ and $y_2(0) = 0$. Simulated free vibration together with the envelope and the instantaneous frequency is shown in Figure 7. The decoupled component after the HT signal decomposition is illustrated in Figure 8. The corresponding backbone (Figure 9(a)) indicates the type of non-linear hardening spring, and small distortion of the damping curve (Figure 9(b)) can be attributed to non-linear effects. The obtained elastic force characteristic (Figure 9(c)) repeats the initial hard spring form of the decoupled subsystem, but the obtained damping force characteristic (Figure 9(d)) differs from the initial linear function due to the decoupling co-ordinate transformation.

5. CONCLUSIONS

We can draw the following conclusions from the analytical signal representation. Whatever the method of non-linear representation, both the instantaneous frequency and

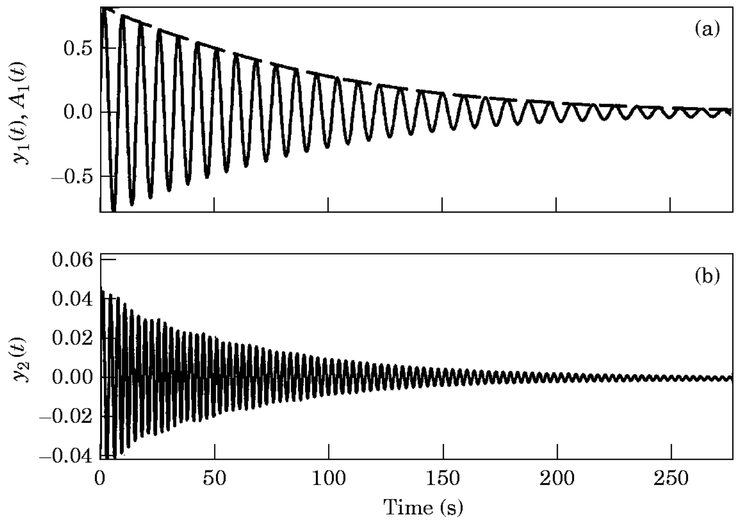

Figure 8. Decoupled free vibration: (a) first component; (b) second component.

the amplitude of free vibration are complicated modulated signals. Non-linear solutions can be represented by an expansion of members with different frequencies or by a time varying signal with oscillated instantaneous frequency and envelope. The instantaneous frequency and envelope of non-linear vibration obtained via the HT are time varying fast oscillating functions. For example, in the presence of a cubic non-linearity and a threefold high harmonics, the frequency of the instantaneous parameter oscillation is twice that the main frequency of vibration. The dependency between the average envelope and the average instantaneous frequency plots the backbone that practically coincides with the theoretical backbone of non-linear vibrations. Using the proposed HT analysis in the time domain we can extract both the instantaneous undamped frequency and also the real non-linear elastic force characteristics.

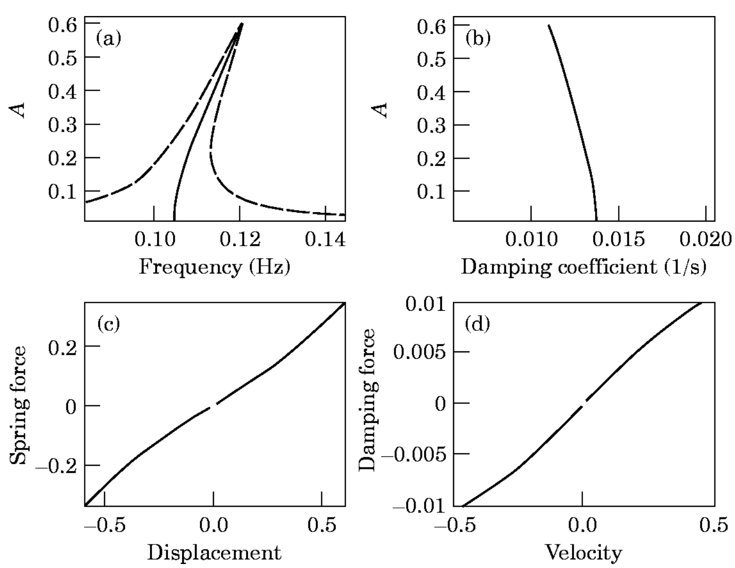

Figure 9. The first component estimated backbone and FRF (a), the damping curve (b), the spring force (c) and the damping force (d) characteristics.

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