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NON-LINEAR SYSTEM VIBRATION ANALYSIS USING HILBERT TRANSFORM—II. FORCED VIBRATION ANALYSIS METHOD 'FORCEVIB'

MICHAEL FELDMAN

Faculty of Mechanical Engineering, Technion—Israel Institute of Technology, Haifa, Israel

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A technique for non-linear system investigation based on the Hilbert transform enables us to identify system instantaneous modal parameters during free vibration analysis and various kinds of excitation of the dynamic system. The direct time domain approach allows the direct extraction of linear and non-linear systems parameters from a measured time signal of input and output. This paper describes forced vibration analysis, and the proposed method determines instantaneous system modal parameters even if an input signal is a high sweep frequency quasi-harmonic signal or random. Some examples of forced vibration analysis of a non-linear system are included.

1. INTRODUCTION

In the recent past a method of studying a dynamic system, based on the Hilbert transform, was proposed for free vibration analysis, where the input signal is an impulse or a shock [1]. The 'FREEVIB' method was suitable for testing linear and non-linear systems and for instantaneous modal parameters identification by free vibration analysis, including the concrete type of non-linear spring and damping characteristics for each mode of the vibratory system. For a great number of real engineering structures with forced quasi-harmonic excitation, it would be useful to consider a new method of modal analysis of non-linear systems with input signal excitation.

2. SYSTEM WITH VISCOUS DAMPING

If we refer to the transformation from equation (2) to equation (3) in part I [1] then, by analogy, we get a differential equation in an analytic signal form for forced vibration of a quasi-linear sdof system with viscous damping

$$\ddot{Y} + 2h_0(A)\dot{Y} + \omega_0^2(A)Y = X/m \tag{1}$$

where $Y(t) = A(t)\,e^{j\psi(t)}$ is a solution of the system, $X(t) = A_x(t)\,e^{j\phi(t)}$ is the forced excitation in the analytical signal form, $h(A)$, $\omega(A)$ is the symmetrical viscous damping and elastic characteristics of the system, m is the mass of the system and ω_0 is the undamped natural frequency. Using the analytic signal form for the system solution and for the excitation equations (1) and (4) in part I [1] we obtain the equation for forced vibration

$$Y\left[\frac{\ddot{A}}{A} - \omega^2 + \omega_0^2 + 2h_0\frac{\dot{A}}{A} + j\left(2\frac{\dot{A}}{A}\omega + \dot{\omega} + 2h_0\omega\right)\right] = X(t)/m \tag{2}$$

where A, ω is the envelope and instantaneous frequency of the solution of the vibratory system.

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2.1. MODAL PARAMETER ESTIMATION

Solving two equations for real and imaginary parts, equation (2), one can write the expression for instantaneous modal parameters as

$$\omega_0^2(t)=\omega^2+\frac{\alpha(t)}{m}-\frac{\beta(t)\dot{A}}{A\omega m}-\frac{\ddot{A}}{A}+\frac{2\dot{A}^2}{A^2}+\frac{\dot{A}\dot{\omega}}{A\omega}$$

$$h_0(t)=\frac{\beta(t)}{2\omega m}-\frac{\dot{A}}{A}-\frac{\dot{\omega}}{2\omega}, \tag{3}$$

where $\omega_0(t)$ is the instantaneous undamped natural frequency of the system, $h_0(t)$ is the instantaneous damping coefficient of the system, ω, A is the instantaneous frequency and envelope (amplitude) of the vibration with their first and second derivatives ($\dot{\omega}, \dot{A}, \ddot{A}$) and $\alpha(t)=\mathrm{Re}\,[X(t)/Y(t)]$, $\beta(t)=\mathrm{Im}\,[X(t)/Y(t)]$ are real and imaginary parts of input and output signals ratio according to the expression

$$\frac{X(t)}{Y(t)}=\alpha(t)+j\beta(t)=\frac{x(t)y(t)+\tilde{x}(t)\tilde{y}(t)}{y^2(t)+\tilde{y}^2(t)}+j\frac{\tilde{x}(t)y(t)-x(t)\tilde{y}(t)}{y^2(t)+\tilde{y}^2(t)} \tag{4}$$

where $x(t)$, $\tilde{x}(t)$ is the force excitation and its Hilbert transform, $y(t)$, $\tilde{y}(t)$ is the vibration of the system and its Hilbert transform.

A formula for modal parameters identification, equation (3), consists of a first and second derivative of the signal envelope and instantaneous frequency, which compensates for transient processes in a dynamic system and determines modal parameters in more complicated testing conditions, for instance when excitation is a non-stationary quasi-harmonic signal with a high sweep frequency. Making a comparison between equation (6) of Part I [1] and the equation for instantaneous modal parameters determination in the case of forced vibration analysis equation (3), it is possible to see that equation (3) is more general, because apart from members with envelope, instantaneous frequency and their derivatives it includes members with input and output signals ratio. When there is no excitation of the system ($\alpha(t)=\beta(t)=0$) equation (3) becomes equal to equation (6) of Part I [1] for instantaneous modal parameters determination in the case of free vibration analysis.

2.2. MODAL MASS VALUE ESTIMATION

As far as equation (3) includes mass value, which is unknown *a priori*, it has first been necessary to define the modal mass value m. In most practical cases the mass value is constant and the natural frequency of the system ω_0 does not vary for a short period of time Δt. Eliminating this natural frequency from equation (3) the mass can be expressed through

$$m=\frac{\Delta\left(\alpha-\dfrac{\beta\dot{A}}{A\omega}\right)}{\Delta\left(-\omega^2+\dfrac{\ddot{A}}{A}-\dfrac{2\dot{A}^2}{A^2}-\dfrac{\dot{A}\dot{\omega}}{A\omega}\right)} \tag{5}$$

where Δ is the deviation of the corresponding functions in the numerator and denominator during time Δt. The use of derivatives for the mass calculation $m=d(\cdot)/d(\cdot)$ because of errors in the third differentiation of experimental data is not recommended.

Representing values of the function in the numerator on the vertical axis and values of the function in the denominator on the horizontal axis for different sets of time points, we get a plot, where the mass given by tan of the slope angle of straight line for a linear sdof system. For a great number of points this mass value calculation could be also done by the least squares method. Thus using the vibration and exciting signals, their Hilbert transforms, first and second derivatives of the vibration we can determine modal parameters of the system according to equation (6) of part I/the present equation (5). It is essential that the instantaneous frequency of an input signal ω have to vary in time, exciting forced vibration with different frequencies.

2.3. STIFFNESS VALUE AND DAMPING FORCE ESTIMATION

Provided that mass value m, natural frequency ω_0 and damping coefficient h_0 are known, stiffness k and damping force c can be calculated as a combination of these functions considering equation (2) of part I

$$k = m\omega_0, \qquad c = 2mh_0. \tag{6}$$

In a linear sdof system with viscous damping the system parameters are constant and do not relate to the amplitude or the frequency of vibration. For a non-linear system all these parameters, except its mass, could depend on the amplitude or (and) the frequency of vibration. Interaction between the natural frequency and the envelope represents a backbone (skeleton curve) of the non-linear system. Interaction between the damping coefficient, the envelope and the forced frequency indicate the type of friction force in the system.

3. SYSTEM WITH STRUCTURAL DAMPING

Let us consider a forced vibration of a quasi-linear sdof system having structural damping or frequency-independent friction

$$\ddot{Y} + \omega_0^2(A)\left[1 + j\frac{\delta(A)}{\pi}\right]Y = X/m \tag{7}$$

where $\delta(A)$ is the logarithmic vibration decrement. Using the system solution and its derivatives in the signal analytic form equation (1) and equation (4) of part I we receive two equations for real and imaginary parts and then get the equations for the natural frequency and damping parameters:

$$\omega_0^2(t) = \omega^2 + \frac{\alpha(t)}{m} - \frac{\ddot{A}}{A} \tag{8}$$

$$\delta(t) = \frac{\pi}{\omega_0^2}\left[\frac{\beta(t)}{\omega m} - \frac{2\dot{A}\omega}{A} - \dot{\omega}\right]$$

where $\omega_0(t)$, $\delta(t)$ are the instantaneous undamped natural frequency and log.decrement of the vibratory system.

Since the mass value is unknown, we first have to estimate the mass eliminating the natural frequency from the first equation, equation (8)

$$m = \frac{\Delta[\alpha(t)]}{\Delta\left(-\omega^2 + \frac{\ddot{A}}{A}\right)}. \tag{9}$$

Comparing equations (3) and (8), we can see that in both models of the vibratory system the instantaneous natural frequency values are virtually equal, because the difference between them is a second-order negligibly small component. Hence there is a certain relation between damping parameters

$$h_0 = \omega_0^2\delta/2\pi\omega, \quad \delta = 2\pi\omega h_0/\omega_0^2 \tag{10}$$

where h_0, δ is the damping coefficient and log.decrement, ω, ω_0 is the instantaneous frequency of the vibration and undamped natural frequency of the system. This shows a way of making distinctions between a frequency-independent and a frequency-dependent friction in the system, tested during forced vibration analysis.

If the damping coefficient of a system h_0 does not vary when we change the frequency of the forced vibration ω, and the decrement δ is directly proportional to this forced frequency, it means that there is some frequency-dependent friction in the system, e.g. a viscous damping. If the decrement δ does not vary during forced frequency changing, and damping coefficient of a system h is inversely proportional to this forced frequency ω, then there is some frequency-independent friction in the system, e.g. a dry friction.

Equations (3) and (8) determine modal parameters of a system as instantaneous functions of time in every point of the process, which makes it possible not only to directly establish non-linear relations between these instantaneous modal parameters, the vibration amplitude and the forced frequency and also to use standard statistical processing procedures, thus obtaining a more precise analysis. For modal parameters estimation we can use either equation (3) or equation (8), as they yield practically the same value of natural frequency, and both or one of the damping parameters can be used for a corresponding type of friction in the system.

4. NON-LINEAR SYSTEM WITH RANDOM EXCITATION

Let us consider a possibility of a linear and a non-linear sdof system identification under a stationary random input signal excitation. Proposed modal analysis equations, including the Hilbert transform and linear transformations are also correct in the case of random input, but their left and right parts of equation (3) have become expressions, which include random values of system parameters and of signals. As far as natural frequency and damping coefficient are random functions, defined through some other random functions (envelope, instantaneous frequency and their derivatives), it is necessary to consider mean values of the system modal parameters.

The mean values of individual sample instantaneous functions from equation (3), when computed by a time average, may be represented by

$$\hat{\omega}_0^2(t) = \overline{\omega^2} + \overline{\frac{\alpha(t)}{m}} - \overline{\frac{\beta(t)\dot{A}}{A\omega m} - \frac{\ddot{A}}{A} + \frac{2\dot{A}^2}{A^2} + \frac{\dot{A}\dot{\omega}}{A\omega}}$$

$$\hat{h}_0(t) = \overline{\frac{\beta(t)}{2\omega m}} - \overline{\frac{\dot{A}}{A} - \frac{\dot{\omega}}{2\omega}}. \tag{11}$$

Let us analyse the right side of equation (11) taking Gaussian random processes as input and output signals on the condition that their instantaneous characteristics (envelope, frequency, phase) are mutually statistically independent [4]. According to equation (1) and equation (4) real and imaginary parts of input and output signal ratios may be defined by

$$\alpha(t) = \frac{A_x(t)}{A(t)} \cos [\phi(t) - \psi(t)], \quad \beta(t) = \frac{A_x(t)}{A(t)} \sin [\phi(t) - \psi(t)].$$

Now, since the mean value of the multiplication of independent random functions is equal to multiplication of their mean values, the last equations can be rewritten as $\bar{\alpha} = \bar{A}_x \overline{\cos} (\phi - \psi)/\bar{A}$ and $\bar{\beta} = \bar{A}_x \overline{\sin} (\phi - \psi)/\bar{A}$. Note that the frequency of forced vibration under a random excitation has symmetric distribution around a resonant frequency, and the phase angle of the system $\phi - \psi = \Delta\phi$ changes from 0 to $-\pi$. Therefore mean values, as trigonometrical functions, are given by

$$\overline{\cos}\,\Delta\phi = \pi^{-1}\int_0^{-\pi}\cos\Delta\phi d(\Delta\phi) = 0, \quad \overline{\sin}\,\Delta\phi = \pi^{-1}\int_0^{-\pi}\sin\Delta\phi d(\Delta\phi) = 2/\pi$$

from which it follows that $\bar{\alpha} = 0$, $\bar{\beta} = 2\bar{A}_x/\pi\bar{A}$.

Mean values of derivatives of envelope and instantaneous frequency of Gaussian random processes are equal to 0 [4], $\bar{\dot{A}} = \bar{\dot{\omega}} = 0$. Thus equation (11) simplifies to

$$\hat{\omega}_0(A) = \bar{\omega}(A), \quad \hat{h}(A) = \bar{A}_x/\pi\bar{A}\omega_0(A)m. \tag{12}$$

This result means in particular, that the backbone of the system under random excitation is a regression curve of an envelope and an instantaneous frequency of forced vibration. For calculation of a damping coefficient we have to average values of envelopes, of a natural frequency and a mass value.

5. MULTI-DEGREE-OF-FREEDOM SYSTEM WITH ONE INPUT

Forced vibration of a mdof system has an important feature: whenever the frequency of excitation is close or equal to one of the undamped natural frequencies, this mode shape is identical to the principal (normal) mode shape of the system. In this case vibration of the system depends only on parameters of this one particular normal mode and on excitation. Using a swept frequency excitation in the range of frequency, including several different natural frequencies of the system, we have conditions where the forced and natural frequencies are very close to each other or coincide at all. In these conditions input and output signals can be used for modal parameters estimation of a corresponding normal mode shape on the base of FORCEVIB procedures. Since the mass of each mode has its own value, the plot of mass estimation equation (5) will be represented by a broken line, where each mass value respectively is given by tan of the slope angle of each line segment. In the case of a mdof system investigation, after calculation of the Hilbert transform and instantaneous functions of signals, it is necessary to use the procedure [equation (5)] for mass value estimation for each mode of the system separately.

6. THE RESULTING EQUATION OF THE METHOD FORCEVIB

It is convenient to limit the solution of the system [equation (2)] and retain only one first quasi-harmonic of the vibration. This assumption means that vibration itself should be a narrow band process and the instantaneous amplitude and frequency of the vibration would be slow functions of time. After low frequency filtration of the instantaneous function the equations for experimental modal analysis equations (3)/(8) become approximate equations. Owing to filtering, it is now possible to represent modal parameters in their traditional form, as a backbone. Accuracy of detection of the system modal parameters as functions of an envelope and an instantaneous frequency would be higher than less negligibly small components of non-linear systems [6]. This accuracy of modal parameter detection corresponds to the accuracy of well known principal approximate analytical methods. In this case, for instance, during scanning frequency of input signals we observe a natural mono-harmonic linearisation of a non-linear system. As a result a higher speed of sweep frequency makes the backbone plot closer to the trivial vertical line.

Direct approximate determination of a backbone (relationship between amplitude and natural frequency), which characterises elastic properties, and a relationship between amplitude and damping characteristics offers the possibility of efficient non-linear system testing avoiding long forced response analysis. Resultant equations of the method are presented in Table 1. The FORCEVIB method uses the vibration signal of system $y(t)$, excitation signal $x(t)$ and includes the next procedures [7]: the Hilbert transform, time derivation, algebraic transforms, least square method, low frequency filtration of the resultant functions, averaging resultant dependencies, involving several individual samples. A proposed method of studying a vibrated system, based on the Hilbert transform, is suitable for testing both linear and non-linear systems, excited by impulse, sweep frequency quasi-harmonic signal, amplitude modulated, bi-harmonic, or random input signal.

7. SIMULATION RESULTS

Let us consider some examples using the FORCEVIB method for the vibratory sdof system modal parameter investigation.

1. *Forced vibration of an elementary linear system.* The equation of forced motion of a linear system is written in the following way: $\ddot{y} + 1.2\dot{y} + (2\pi 10)^2 y = 400\sin(2\pi 4t^2)$ where, on the left, we have a system with natural frequency $f_0 = 10$ Hz, damping coefficient $h_0 = 0.6\ \mathrm{s}^{-1}$, and on the right, force excitation with high speed of sweep frequency $\dot{f} = 4$ Hz/s. Figure 1 shows the force [Fig. 1(a)] and the system time history [Fig. 1(b)], which in the case of non-stationary excitation have some pulsations. Nevertheless the obtained modal parameters of the system tested do not have errors or distortions in practice [Fig. 1(c), (d)].

TABLE 1

Resultant equations for forced vibration identification

Instantaneous characteristics	Equations	Dimension
Amplitude	$A = \sqrt{y^2 + \tilde{y}^2}$	m
Forced vibration frequency	$f = \dfrac{y\dot{\tilde{y}} - \tilde{y}\dot{y}}{2\pi(y^2 + \tilde{y}^2)}$	Hz
Mass value (viscous damping)	$m_1 = \dfrac{\Delta[(x\dot{\tilde{y}} - \tilde{x}\dot{y})(y\dot{\tilde{y}} - \tilde{y}\dot{y})]}{\Delta[(y\dot{\tilde{y}} - \tilde{y}\dot{y})(\ddot{\tilde{y}}\dot{y} - \ddot{y}\dot{\tilde{y}})]}$	Ns^2/m
Mass value (structural damping)	$m_2 = \dfrac{\Delta[(xy + \tilde{x}\tilde{y})(y^2 + \tilde{y}^2)]}{\Delta[(y^2 + \tilde{y}^2)(-y\ddot{y} - \tilde{y}\ddot{\tilde{y}})]}$	Ns^2/m
Undamped natural frequency (viscous damping)	$f_{01} = 0.5\pi^{-1}\left[\dfrac{(x\dot{\tilde{y}} - \tilde{x}\dot{y})/m - \ddot{\tilde{y}}\dot{y} - \ddot{y}\dot{\tilde{y}}}{y\dot{\tilde{y}} - \tilde{y}\dot{y}}\right]^{1/2}$	Hz
Undamped natural frequency (structural damping)	$f_{02} = 0.5\pi^{-1}\left[\dfrac{(xy + \tilde{x}\tilde{y})/m - y\ddot{y} - \tilde{y}\ddot{\tilde{y}}}{y^2 + \tilde{y}^2}\right]^{1/2}$	Hz
Damping coefficient	$h_0 = 0.5\dfrac{(\tilde{x}y - x\tilde{y})/m + \tilde{y}\ddot{y} - y\ddot{\tilde{y}}}{y\dot{\tilde{y}} - \tilde{y}\dot{y}}$	s^{-1}
Logarithmic decrement	$\delta = \pi\dfrac{(\tilde{x}y - x\tilde{y})/m + \tilde{y}\ddot{y} - \ddot{\tilde{y}}}{(xy + \tilde{x}\tilde{y})/m - y\ddot{y} - \tilde{y}\ddot{\tilde{y}}}$	—

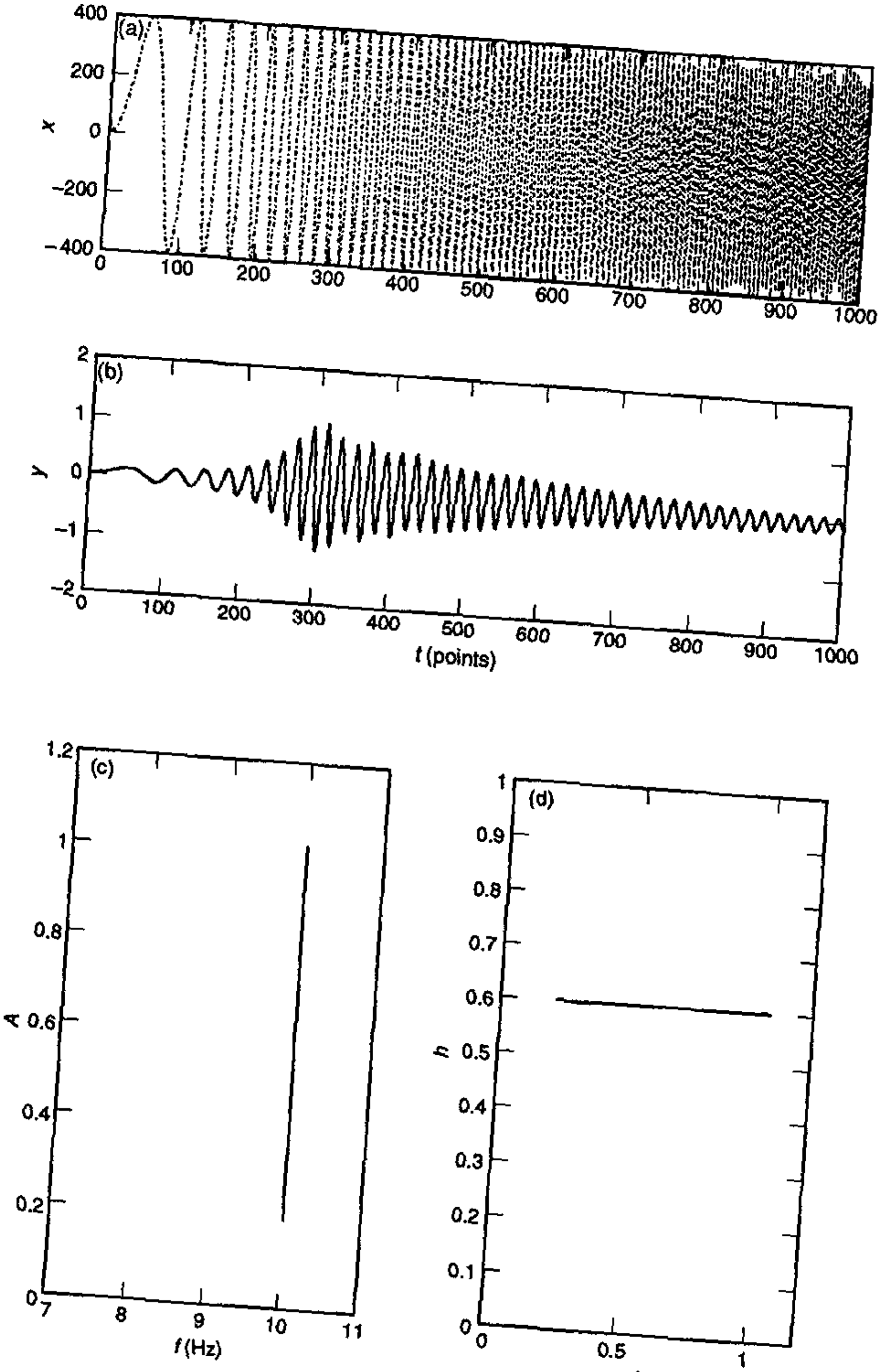

Figure 1. Forced vibration linear system identification.

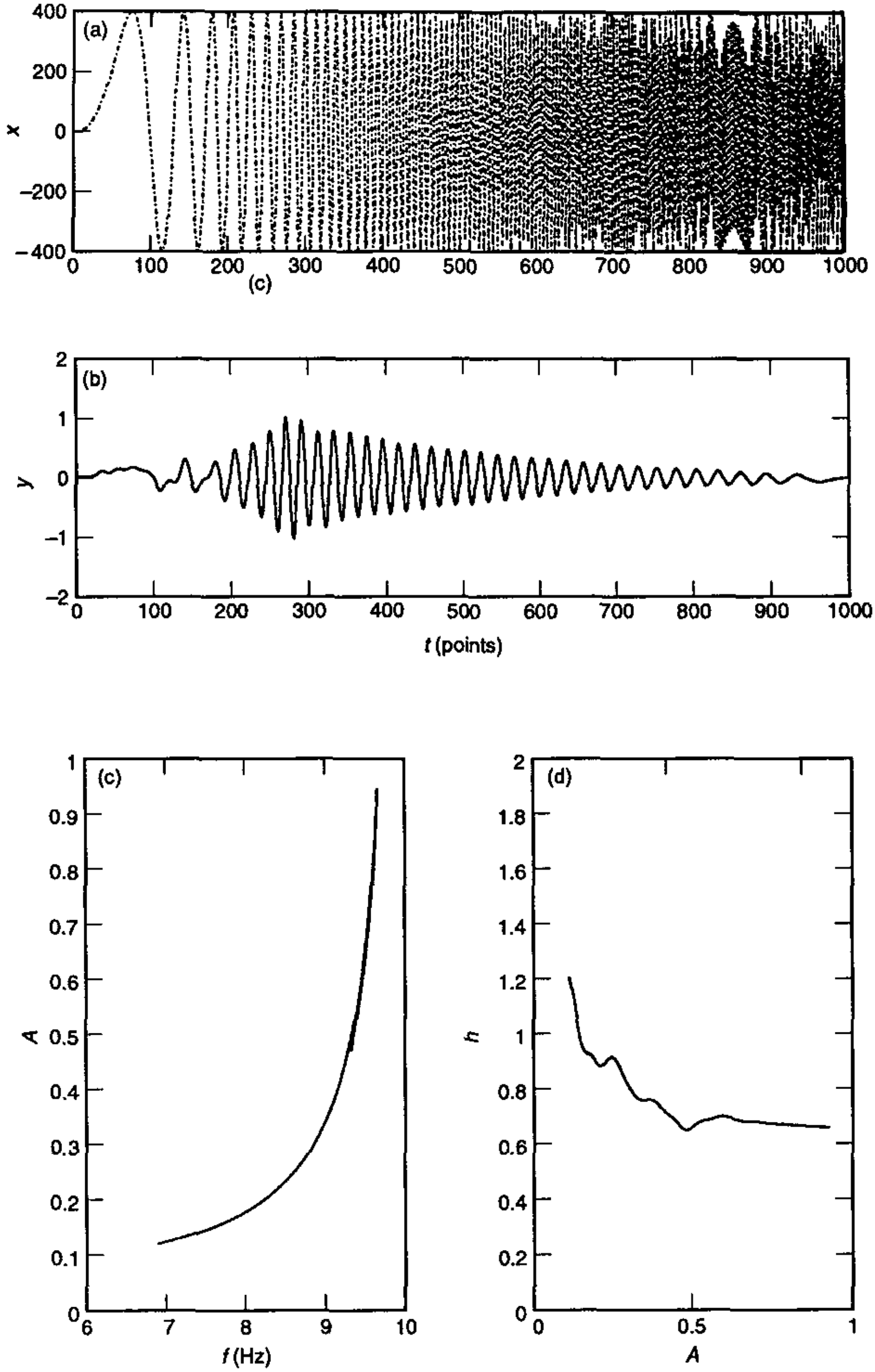

Figure 2. Backlash and dry friction system identification using forced vibration.

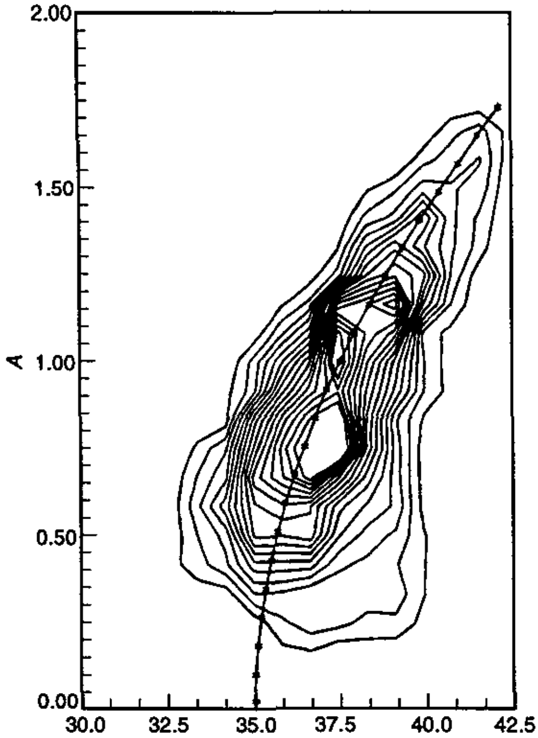

Figure 3. Backbone of the Duffing system under random excitation: *, theoretical backbone.

2. *Forced vibration of the non-linear system with a dry friction and a backlash.*

$$\ddot{y} + 1.2\dot{y} + 5\,\text{sign}\,(\dot{y}) + (2\pi 10)^2[y - 0.05\,\text{sign}\,(y - 0.05)] = x \quad \text{if } |y| > z$$
$$\ddot{y} + 1.2\dot{y} + 5\,\text{sign}\,(\dot{y}) = x \quad \text{if } |y| \leqslant z$$
$$y_0 = 2, \dot{y}_0 = 0$$

where right-hand side is $x(t) = 300 \sin (2\pi 3t^{2.5})$. The backbone of the system obtained after its identification using FORCEVIB procedures virtually coincides with the theoretical skeleton curve [Fig. 2(c)]. It is a monotonous increasing curve line which has a trivial vertical line of linear system as an asymptote on the right and shows a clearance value on the amplitude axis where the natural frequency comes nearer to zero. The plot of the dependence of the instantaneous damping coefficient and envelope has some deviations in a small amplitude zone over the very high speed of sweep frequency, but it has a monotonous decreasing hyperbola form [Fig. 2(d)] which leads us to the conclusion that in this case Coulomb friction is operated in the system.

3. *A non-linear Duffing system forced by Gaussian white noise* $\ddot{y} + 20\dot{y} + (70\pi)^2 y(1 + 0.2y^2) = \xi(t)$.

The following instantaneous modal parameters of the system have been computed and are shown in Fig. 3. The backbone obtained for the regression of the instantaneous natural frequency and envelope practically coincides with the theoretical backbone (**,*) [5] of the tested system.

8. SUMMARY AND CONCLUSION

In conclusion, it can be stated that an interesting and promising experimental method for identification of non-linearities in stiffness and damping characteristics of a vibration system has been developed. The method is based on input and output time domain measurements and on their Hilbert transforms. The method defines instantaneous modal parameters (backbones, damping dependencies) of a system under a slow or a very fast swept frequency test, narrow or wide band random excitation.

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