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NON-LINEAR SYSTEM VIBRATION ANALYSIS USING HILBERT TRANSFORM—I. FREE VIBRATION ANALYSIS METHOD 'FREEVIB'

MICHAEL FELDMAN

Faculty of Mechanical Engineering, Technion—Israel Institute of Technology, Haifa, Israel

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This study concentrates on techniques for non-linear system investigation, which enable us to identify instantaneous modal parameters (natural frequencies, damping characteristics and their dependencies on a vibration amplitude and frequency) in the process of free vibration analysis and through various kinds of excitation of the dynamic system. Modal parameters identification is based on input and output signal measurements from the dynamic system and on signal processing, including the Hilbert transform. Part I presents the free vibration analysis of the system including the identification technique for the main characteristics of the non-linear oscillatory systems. For non-linear effects representation, a sdof system with a hard spring has been considered.

1. INTRODUCTION

An experimental analysis of free vibration with impulse (shock) excitation of the dynamic system is often used for estimation of modal parameters of mechanical structures [1]. Impulse input signals are suitable for a wide variety of engineering structures, and the test is very fast and particularly convenient in the laboratory. There are limitations to known vibration analysis methods: only linear dynamic systems can be tested through frequency response functions, and low accuracy of the determination of the non-linear dissipation characteristics are reached by the small number of peak points in the damping process. There is also a method of free vibration investigation of non-linear systems using the Hilbert transform [2]. However this method helps only in cases of determination of an average damping coefficient by taking logarithms. It is of no use for instantaneous dissipation characteristics determination.

In this paper a new method for studying a dynamic system is proposed based on the Hilbert transform it is suitable for both linear and non-linear system testing during input impulse excitation. The method has some advantages and is recommended for instantaneous modal parameters identification, including determination of concrete type non-linear spring and damping characteristics of quasi-linear vibratory systems by free vibration analysis.

2. THEORETICAL BASES

According to analytic signal theory a large number of processes including vibration of the system $y(t)$ can be converted by the Hilbert transform to a new function and also

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represented in the form of the combination of slow varying functions, called envelope and instantaneous phase [3]:

$$Y(t) = y(t) + j\tilde{y}(t) = A(t)\exp[j\psi(t)]$$
$$y(t) = A(t)\cos\psi(t), \qquad A(t) = \sqrt{y^2(t) + \tilde{y}^2(t)}$$
$$\psi(t) = \arctan[\tilde{y}(t)/y(t)], \tag{1}$$

where $y(t)$, $\tilde{y}(t)$ is vibration and its Hilbert transform (real valued functions), $Y(t)$ is vibration in an analytic signal form (complex time-function), $A(t)$, $\psi(t)$ is an envelope signal (amplitude) and an instantaneous phase (real valued functions). The function $\tilde{y}(t)$ is defined as the Hilbert transform of $y(t)$ [3]:

$$\mathbf{H}[y(t)] = \tilde{y}(t) = \frac{1}{\pi t} * y(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y(\tau)}{t-\tau}\,d\tau$$

The Hilbert transform can be considered to be a filter which simply shifts phases of all frequency components of its input by $-\pi/2$ radians. In order to convert the complex time-function of the analytic signal to its original function it is necessary to use the substitution $y(t) = (Y(t) + Y^*(t))/2$. The instantaneous angular frequency is the time-derivative of the instantaneous phase:

$$\omega(t) = \dot{\psi}(t) = \frac{y(t)\dot{\tilde{y}}(t) - \dot{y}(t)\tilde{y}(t)}{A^2(t)} = \mathrm{Im}\left[\frac{\dot{Y}(t)}{Y(t)}\right]$$

We note that the time-derivative of the amplitude is also very important for analytic signal representation:

$$\dot{A}(t) = \frac{y(t)\dot{y}(t) + \tilde{y}(t)\dot{\tilde{y}}(t)}{A(t)} = A(t)\,\mathrm{Re}\left[\frac{\dot{Y}(t)}{Y(t)}\right].$$

Let us consider free vibration of a quasi-linear sdof system with viscous damping

$$\ddot{y} + 2h_0(A)\dot{y} + \omega_0^2(A)y = 0 \tag{2}$$

where y is a system solution, $h_0(A) = c(A)/2m$ are the symmetrical viscous damping characteristics, $\omega_0^2(A) = k(A)/m$ is the undamped natural frequency, m is the mass of the system and $k(A)$ is the symmetric elastic characteristics of the system.

According to the main properties of the Hilbert transform, where $h(t)$ and $y(t)$ are signals with non-overlapping spectra [4], where $h(t)$ is lowpass and $y(t)$ is highpass, then $\mathbf{H}[h(t)y(t)] = h(t)\mathbf{H}[y(t)]$, $\mathbf{H}(\dot{y}) = \dot{\tilde{y}}$, and we can use the Hilbert transform for both sides of equation (2). Multiplying each side of the obtained new equation by j and adding it to the corresponding side of equation (2) we get a differential equation for the analytic signal

$$\ddot{Y} + 2h_0(A)\dot{Y} + \omega_0^2(A)Y = 0 \tag{3}$$

where Y is a system solution in the analytic signal form.

Using the analytic signal form equation (1), together with its two first derivatives

$$\dot{Y} = Y(t)[\dot{A}(t)/A(t) + j\omega(t)]$$
$$\ddot{Y} = Y(t)[\ddot{A}(t)/A(t) - \omega^2(t) + 2j\dot{A}(t)\omega(t)/A(t) + j\dot{\omega}(t)] \tag{4}$$

to solve the dynamic system equation (3) we get the equation for free vibration analysis

$$Y\left[\frac{\ddot{A}}{A} - \omega^2 + \omega_0^2 + 2h_0\frac{\dot{A}}{A} + j\left(2\frac{\dot{A}}{A}\omega + \dot{\omega} + 2h_0\omega\right)\right] = 0 \tag{5}$$

where A, $\omega = \dot{\psi}$-envelope and instantaneous frequency of the vibratory system solution.

Solving two equations for real and imaginary parts of equation (5), we can write the expression for instantaneous modal parameters as functions of a first and a second derivative of the signal envelope and the instantaneous frequency.

$$\omega_0^2(t) = \omega^2 - \frac{\ddot{A}}{A} + \frac{2\dot{A}^2}{A^2} + \frac{\dot{A}\dot{\omega}}{A\omega}, \qquad h_0(t) = -\frac{\dot{A}}{A} - \frac{\dot{\omega}}{2\omega}, \tag{6}$$

where $\omega_0(t)$ is the instantaneous undamped natural frequency of the system, $h_0(t)$ is the instantaneous damping coefficient of the system, ω, A is the instantaneous frequency and envelope (amplitude) of the vibration with their first and second derivatives ($\dot{\omega}, \dot{A}, \ddot{A}$).

By analogy with the differential equation (3) the equation of motion of the system with structural damping independent of the vibration frequency will be

$$\ddot{Y} + \omega_0^2(A)\left[1 + j\frac{\delta(A)}{\pi}\right]Y = 0 \tag{7}$$

where $\delta(A)$ is logarithmic vibration decrement.

Substituting the analytic signal form equation (1) and its two first derivatives equation (4) into the equation of motion equation (7) we get

$$\omega_0^2(t) = \omega^2 - \frac{\ddot{A}}{A}, \qquad \delta(t) = -\frac{2\pi\dot{A}\omega}{A\omega_0^2} - \frac{\pi\dot{\omega}}{\omega_0^2}, \tag{8}$$

where $\omega_0(t)$ is the instantaneous undamped natural frequency of the system and $\delta(t)$, is the instantaneous logarithmic vibration decrement.

A comparison between equations (6) and (8) shows that in both models the instantaneous natural undamped frequency is practically equal to the instantaneous free vibration frequency, because the difference in the case of small damping is of second-order negligibly small components. These two equations determine the undamped natural frequency and damping parameters of the system as instantaneous functions of time in every point of the vibration process. It points out the direct way for establishing non-linear relations between instantaneous modal parameters and the vibration amplitude and also for using standard statistical processing procedures, thus making the system modal analysis more precise.

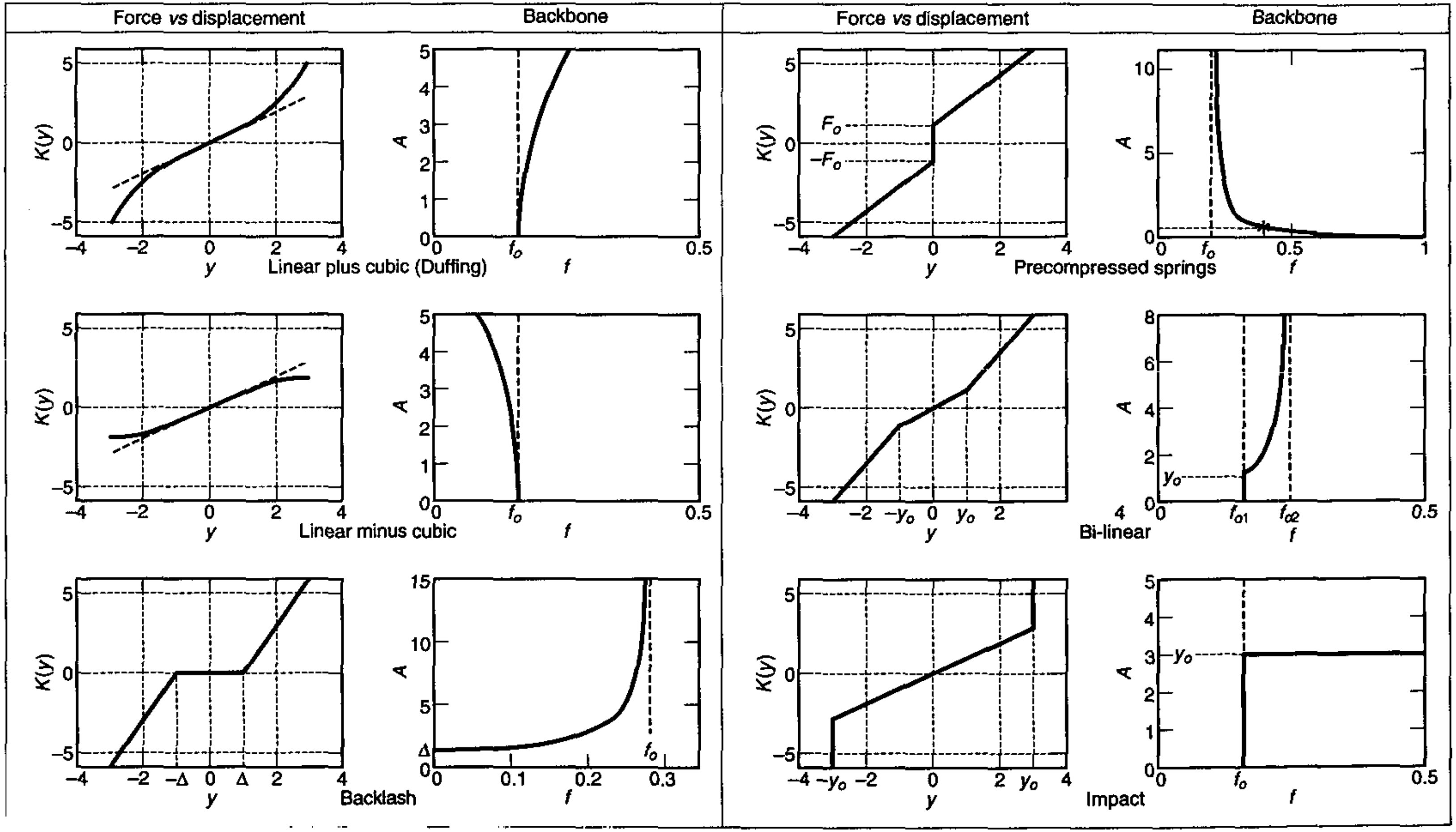
3. MAIN CHARACTERISTICS OF THE NON-LINEAR OSCILLATORY SYSTEM

The instantaneous damping coefficient can be determined by the form of the symmetric dissipative function and the variation of the instantaneous natural frequency in time by the symmetric elastic restoring forces. In general when investigating non-linear systems the instantaneous damping coefficient and the natural frequency become functions of the amplitude. In the particular case of a linear system the instantaneous natural frequency and the instantaneous damping coefficient or decrement do not vary in time.

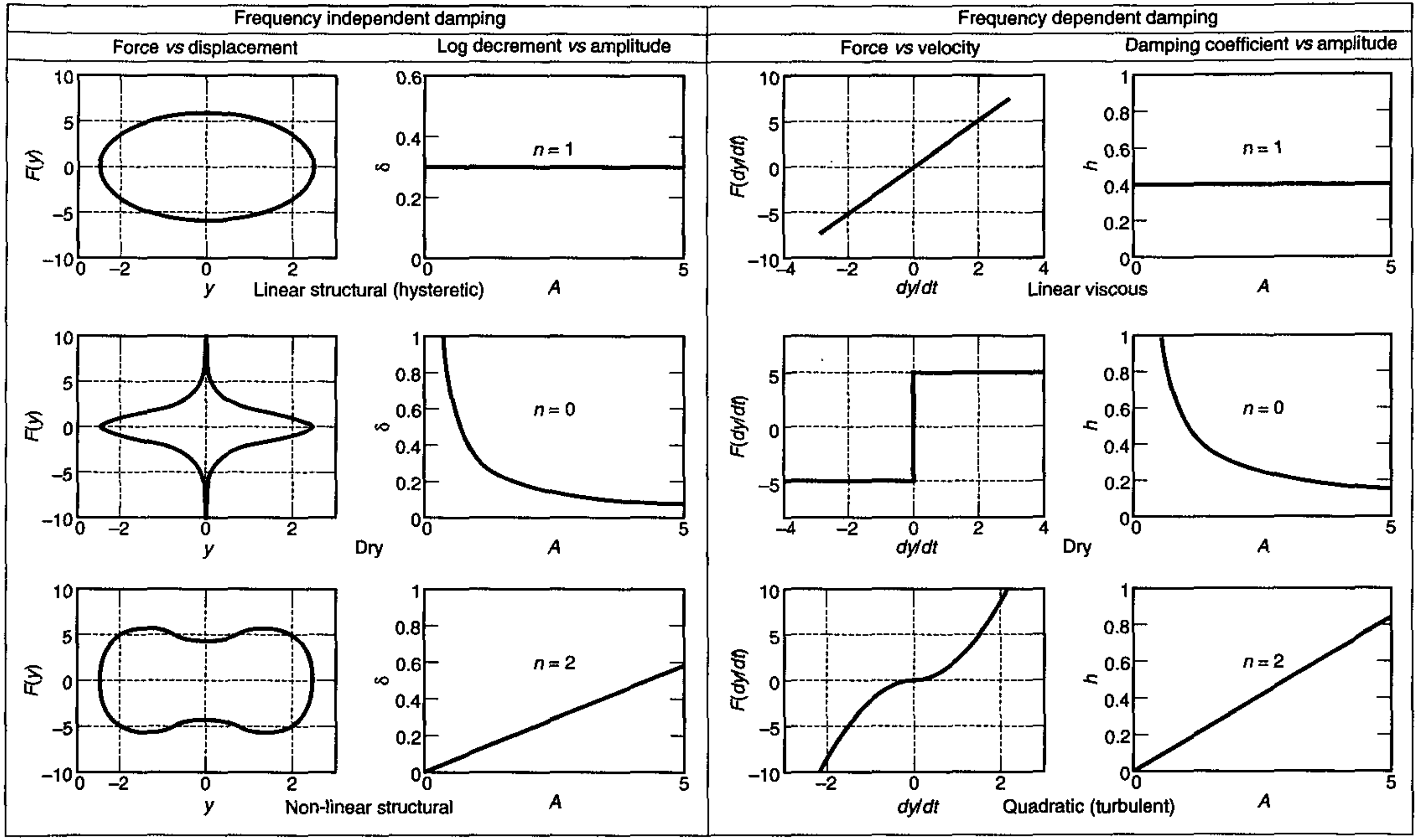
Let us consider some typical cases of interactions between the non-linear system characteristics. In most cases if the system to be tested has non-linear elastic forces, the natural frequency will depend decisively on the amplitude of vibrations. We present this departure from synchronism of the vibration in the form of a regression curve of the instantaneous amplitude on the instantaneous frequency which represents a kind of skeleton curve (or backbone) of the system under test. Every typical non-linearity in the spring (Duffing system, backlash or clearance, pretensioned system, bi-linear system, impact system etc.) has its unique form of skeleton curve. Several possible forms of $A(\omega_0)$ are shown in Table 1.

TABLE 1

Typical non-linear spring models with symmetric characteristics



Typical non-linear damping models with symmetric characteristics



Analysis of the topography of the skeleton curve is essential for evaluation of the properties of the particular vibrating system, e.g., through reconstruction of any characteristics of the elastic forces (Table 1). For example, a skeleton curve of the system with backlash is a monotonic increasing curve which has a trivial vertical line of linear system as an asymptote on the right side and cuts off a clearance value on the amplitude axis on the left.

If non-linear dissipative forces are operating in the test system, the values obtained from *the instantaneous vibration damping coefficient may depend on the instantaneous* amplitude (Table 2). Coulomb or dry friction in particular has a plot of damping coefficient and envelope dependence as a monotonic decreasing hyperbola. Experimental studies of the vibration of engineering structures indicate that the nature of dissipative forces is such that the frequency has practically no influence on the value of logarithmic decrement, and that a model of frequency-independent friction should be used to describe the vibrations.

4. THE RESULTING EQUATION OF THE METHOD 'FREEVIB'

The proposed method for analysis of the machine vibration offers a way of directly plotting the system skeleton curve, which includes modal frequency and non-linearity in spring characteristics, as well as dependencies between damping parameters and the amplitude, which contains modal damping together with non-linearity in friction. The method is suitable for efficient oscillatory system testing avoiding time-eating forced response analysis. Resultant equations for the 'FREEVIB' method are presented in Table 3.

There are two ways of implementation of a free vibration process in the system tested: either suddenly switch off the exciter during vibration resonant sinusoidal testing, or use the shock excitation and a narrow band filter to get a response in the area around the natural frequency. Free vibration analysis means, that we deal only with vibration $y(t)$ when the excitation signal has ceased.

Free vibration of a quasi-linear vibratory system is a narrow band process, so it is possible to ignore the second-order negligible small components in equation (6). The result

TABLE 3

Resultant equations for free vibration identification

Instantaneous characteristics	Equations	Dimension
Amplitude	$A=\sqrt{y^2+\tilde{y}^2}$	m
Free vibration frequency	$f=\dfrac{y\dot{\tilde{y}}-\tilde{y}\dot{y}}{2\pi(y^2+\tilde{y}^2)}$	Hz
Undamped natural frequency (viscous damping)	$f_{01}=0{\cdot}5\pi^{-1}\left(\dfrac{\ddot{\tilde{y}}\dot{y}-\ddot{y}\dot{\tilde{y}}}{y\dot{\tilde{y}}-\tilde{y}\dot{y}}\right)^{1/2}$	Hz
Undamped natural frequency (structural damping)	$f_{02}=0{\cdot}5\pi^{-1}\left(\dfrac{-y\ddot{y}-\tilde{y}\ddot{\tilde{y}}}{y^2+\tilde{y}^2}\right)^{1/2}$	Hz
Damping coefficient	$h_0=0{\cdot}5\,\dfrac{\tilde{y}\ddot{y}-y\ddot{\tilde{y}}}{y\dot{\tilde{y}}-\tilde{y}\dot{y}}$	s^{-1}
Logarithmic decrement	$\delta=\pi\,\dfrac{\tilde{y}\ddot{y}-y\ddot{\tilde{y}}}{-y\ddot{y}-\tilde{y}\ddot{\tilde{y}}}$	—

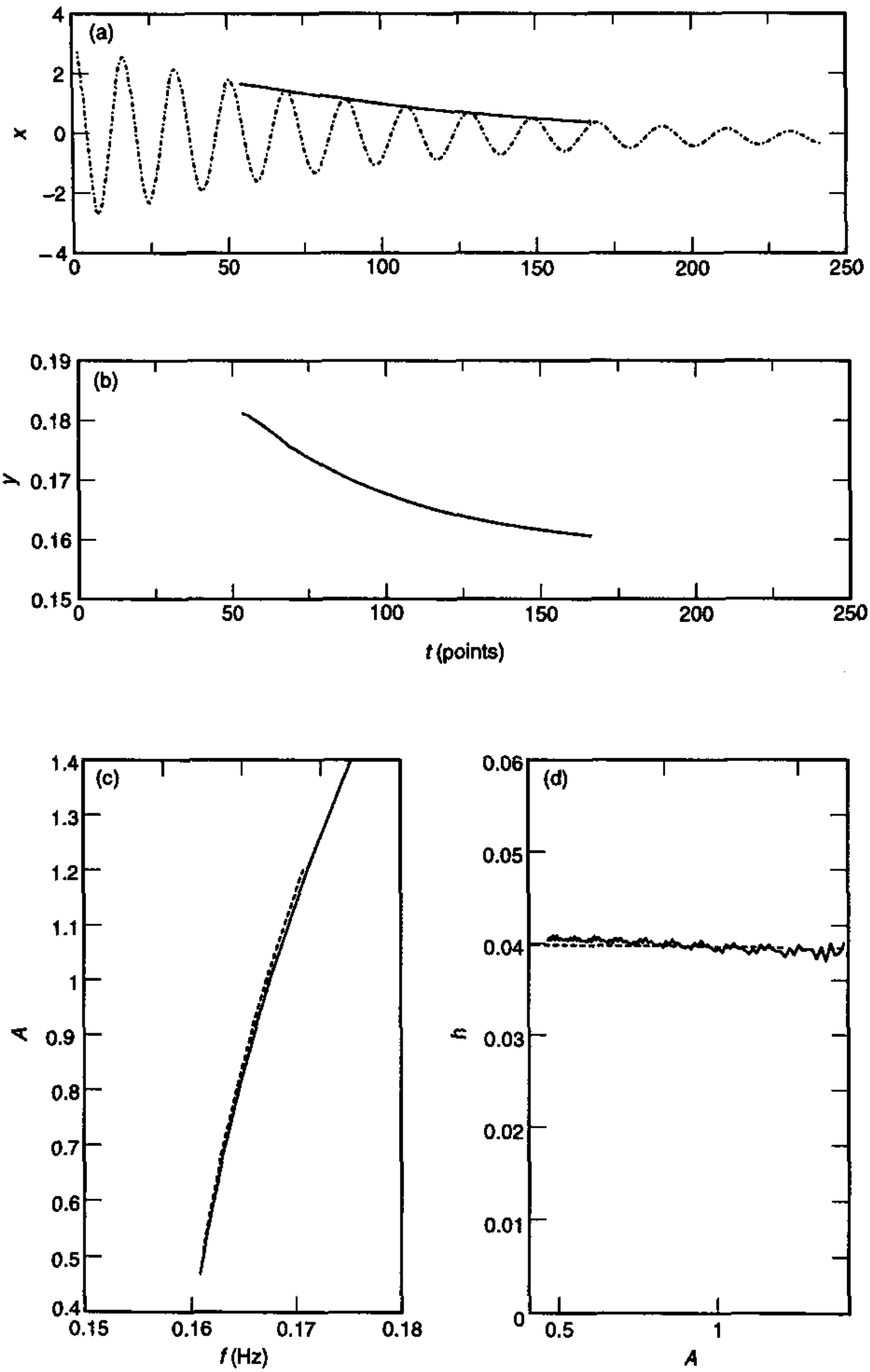


Figure 1. Duffing system identification using free vibration.

becomes

$$\omega_0(t) \cong \omega(t), \qquad h_0(t) = -\frac{\dot{A}}{A} - \frac{\dot{\omega}}{2\omega}, \qquad \delta(t) \cong 2\pi h_0/\omega. \tag{9}$$

Equation (9) shows that the natural frequency of the system is practically equal to its instantaneous frequency while the damping coefficient is assumed to be proportional to the first power of the velocity of the envelope and the instantaneous frequency.

It is clear now that the average damping coefficient $\bar{h}$ in linear system ($\dot{\omega} = 0$), calculated as integral

$$\bar{h} = T^{-1} \int_0^T h(t) t = T^{-1} \int_0^T \frac{\dot{A}(t)}{A(t)} t = T^{-1} \ln \frac{A(t)}{A_0},$$

is equal to the logarithm of the envelope.

'FREEVIB' uses a free vibration signal after narrow band filtration and consists of the following procedures [6]: the Hilbert transform, time derivative, algebraic transforms, low frequency filtration of the resultant functions, and averaging resultant dependencies involving several individual samples. From the expressions in the Table 3 it follows, that the instantaneous modal frequency and damping parameters are functions of time and can be determined at any point of the damped process. The total number of these points which map the free vibration is much greater than that of the peak points of the process. It opens the way for establishing non-linear relations between instantaneous functions and for using statistical processing procedures, making the analysis more precise. To get the best result, it is useful to filter and average a few free vibration backbones and damping-envelope dependencies of the system tested.

5. NON-LINEAR DUFFING EQUATION ANALYSIS

As an example, let us consider the free vibration and instantaneous characteristics of the process in the elementary system with non-linear hard spring $\ddot{y} + 0.08\dot{y} + y(1 + 0.14y^2) = 0$, $y_0 = 3$, $\dot{y}_0 = 0$, which were computer calculated for 300 points at 0.3-s intervals. The use of digital filtration programs for the Hilbert transform with less than 1% non-uniformity of frequency characteristics ensured high accuracy of calculation from the initial process [· – · –, Fig. 1, (a)] the instantaneous amplitude [———, Fig. 1, (a)] and the frequency [Fig. 1, (b)]. The system backbone [Fig. 1, (c)] obtained after identification and using the above procedures practically coincides with the theoretical skeleton curve [5] $A \approx 3(\omega_0^2 - 1)^{1/2}$.

The plot of the dependence of the instantaneous damping coefficient and envelope having some small deviations but practically constant value of the damping coefficient [Fig. 1, (d)] points to the conclusion that in this case we are dealing with viscous linear damping performing within the system, and that the reconstructed value of the damping coefficient is equal to the model value ($h = 0.04$).

6. SUMMARY AND CONCLUSION

As non-linear dissipative and elastic forces have totally different effects on free vibration (energy dissipation lowers the instantaneous amplitude, while non-linear elasticity links the instantaneous frequency and amplitude in a certain relationship) it is possible to determine some aspects of the behaviour of these forces. For this identification we propose that relationships be constructed between the instantaneous frequency and amplitude plus curves of the instantaneous decrement as a function of amplitude. The identification technique developed here should be of value in many areas of mechanical oscillatory systems having various features of non-linear behaviour.

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