## ARTICLE

## LARGE FREQUENCY RANGE EASILY TUNABLE DIGITAL NOTCH FILTER <br> by Jean-Paul Sandoz

Notch filters have always played an important role in communication systems, signal detection, signal estimation and many other applications. With the ever growing problem of non-stationary interference situations, it is natural to design and implement digital notch filters whose frequency can be easily and quickly changed so as to match particular conditions in real-time. Very often, it is also highly desirable to introduce as little additional delay as possible, thus the infinite impulse response (IIR) type digital filter has been selected. IIR based structures are also well known for their ability to achieve very good rejection over a predetermined bandwidth with a relatively low order (typically less than 10). The most important characteristics of the proposed solution are the following:

- Cascade of second order notch filter sections
- Maximum to minimum stopband frequency ratio: 20
- Fast filter stop-band frequency modification.
- One multiplication per second order section.
- Good stop-band width to transition band ratio.
- Low group delay.

Typical design example:

- Clock frequency (fs): 10 kHz
- Stop-band frequency tuning range: $50 \mathrm{~Hz} \rightarrow 2000 \mathrm{~Hz}$
- $\quad-3 \mathrm{~dB}$ stopband width: 38 Hz
- -40 dB stopband width: 10 Hz
- Transition band: 14 Hz
- Number of second order sections: 9
- 2 kHz to 50 Hz transient time: 80 ms
- Rejection bandwidth to transition band ratio: 0.71

The following definitions are used (Fig.1):
-3 dB Stop-Band Width: $\mathrm{Bw}_{-3 \mathrm{~dB}}=$ freq. $4-$ freq. 1
Stop-Band Width: $\quad \mathrm{Bw}_{\text {Goal }}=$ freq. $3-$ freq. 2
Transition Band:

$$
\begin{aligned}
\mathrm{B}_{\text {tran }} & =\text { freq. } 2-\text { freq. } 1 \\
& \approx \text { freq. } 4-\text { freq. } 3
\end{aligned}
$$

Transition to Stop-Band Ratio: $B_{\text {tran }} /{B w_{G o a l}}=\alpha$
Stop-Band Attenuation in $\mathrm{dB}: \quad \mathrm{Att}_{\mathrm{dB}}=20 \log (\mathrm{Att})$

The design is based on the equivalence between analog and digital filters. Thus, the second order digital notch filter form used is the following:

$$
\mathrm{H}_{\mathrm{REJ}}(\mathrm{z})=1-\mathrm{K} \cdot \frac{1-\mathrm{z}^{-2}}{1+\mathrm{W} \cdot \mathrm{z}^{-1}+\mathrm{b} 2 \cdot \mathrm{z}^{-2}}
$$

Where: $\mathrm{b} 2=1-2 \cdot \mathrm{~K}$
With this solution, only one single parameter $\boldsymbol{W}$ adjusts the notch filter frequency, while the stop-band or rejection bandwidth at $-3 \mathrm{~dB}\left(B w_{3 d B}\right)$ is determined by K and b2. It is worthwhile mentioning that $B w_{3 d B}$ is constant across the full operating frequency range. In addition, a reasonably small $\alpha$ value is achieved by cascading a number of $L$ evenly spaced (i.e fo $-2 \Delta f$, fo $\Delta \mathrm{f}$, fo, fo $+\Delta \mathrm{f}$, fo $+2 \Delta \mathrm{f}$ ) identical structures with $\Delta \mathrm{f}$ been the frequency interval between consecutive notch filter center frequencies. Therefore, the transfer function of any individual filter has the following form:

$$
\operatorname{HREJn}(\mathrm{z})=1-K \cdot \frac{1-z^{-2}}{1+(W+(n-m) \cdot \Delta W) \cdot z^{-1}+b 2 \cdot z^{-2}}
$$

With $\mathrm{m}=(\mathrm{L}-1) / 2$ i.e. $\mathrm{L}=5 \Rightarrow \mathrm{~m}=2$ and $\mathrm{n}: 0,1,2,3,4$
Thus, the complete transfer function has can be written as:
$H_{\text {REJtot }}(z)=\prod_{n=0}^{L-1}\left[1-K \cdot \frac{1-z^{-2}}{1+(W+(n-m) \cdot \Delta W) \cdot z^{-1}+b 2 \cdot z^{-2}}\right]$

The stop-band width $\left(B w_{\text {Goal }}\right) \quad$ is the band in which the attenuation is at least equal to the factor $A t t$, or $20 \log$ (Att) in dB . After somewhat involved algebraic computations, it can be shown that $B w_{\text {Goal }}$ is independent of $W$ and it can be computed as follows:

$$
\mathrm{Bw}_{\text {Goal }} \approx \mathrm{fs} \cdot 2^{-\mathrm{h}-1} /(\pi \mathrm{Att}), \mathrm{K}=2^{-\mathrm{h}-1}, \mathrm{~h}>4
$$

The tuning parameter " W " can be computed exactly with this expression:

$$
\mathrm{W}=-2 \cdot\left(1-2^{-h-1}\right) \cdot \cos \left(\frac{2 \cdot \pi \cdot \mathrm{fo}}{\mathrm{fs}}\right)
$$

fs: Sampling frequency, fo: Notch filter center frequency

Then, $\Delta \mathrm{W}$ can be approximated as follows:
$\Delta \mathrm{W}(\mathrm{fo}) \approx 8 \cdot \Delta \mathrm{f} \cdot \pi^{2}$ fo $/ \mathrm{fs}^{2}$ with $\mathrm{fo}<0.2 \mathrm{fs}$
An example with $\mathrm{h}=9$ and $\Delta \mathrm{f}=5.3 \mathrm{~Hz}$ is shown on Fig. 2.
From all the previous relationships, an 'Easily Tunable Notch Filter" design guideline has been derived.

## Step 1: Design Objectives:

$$
\mathrm{fs}, \mathrm{fo}, \mathrm{Att}^{\left(\mathrm{Bw}_{\text {Goal }} \text { and } \alpha\right.}
$$

## Step 2: h, b2 and K

$$
\begin{aligned}
& \mathrm{h}_{\text {esti }} \log _{2}\left[\frac{\mathrm{fs} \cdot \sqrt{\mathrm{Att}}}{\mathrm{Bw} \mathrm{Goal}^{\prime}(1+2 \cdot \alpha)^{2} \cdot 2 \cdot \pi}\right] \\
& \mathbf{h}_{\text {esti }} \rightarrow \mathbf{h} \text { (nearest integer) } \\
& \mathbf{b 2}=\mathbf{2}^{-\mathbf{h}} \quad \mathbf{K}=\mathbf{2}^{-\mathbf{h}-\mathbf{1}}
\end{aligned}
$$

## Step 3: L

$$
\begin{aligned}
& \mathrm{L}_{\text {esti }}=\left[\frac{\mathrm{Bw}_{\mathrm{Goal}} \cdot(1+2 \cdot \alpha) \cdot \pi}{\mathrm{fs} \cdot 2^{-\mathrm{h}-1}}\right]^{2} \\
& \mathbf{L}_{\text {esti }} \Rightarrow \mathbf{L} \text { (nearest odd integer) }
\end{aligned}
$$

## Step 4: $\Delta \mathrm{W}, \mathrm{W}$

$$
\begin{gathered}
\mathrm{W}=-2 \cdot\left(1-2^{-\mathrm{h}-1}\right) \cdot \cos \left(\frac{2 \cdot \pi \cdot \mathrm{fo}}{\mathrm{fs}}\right) \\
\Delta \mathrm{W}=\frac{\pi \cdot 2^{-h+2} \cdot \mathrm{fo}}{\sqrt{\mathrm{Att} \cdot f s}}
\end{gathered}
$$

## Notes:

- The relations proposed for the computation of both $\mathrm{h}_{\text {esti }}$ and $\mathrm{L}_{\text {esti }}$ are approximations. Thus, it is suggested to check the notch filter overall performances with all the combinations of adjacent values (e.g. h,L: 6,5 ; 6,7;7,5 and 7,7).
- From the above relations, it can be observed that if a relatively small transition band compared to the rejection band is required (i.e. $\alpha \ll 1$ ), that will automatically increase $h$ which in turn will increase $L$. Thus, trade-offs will always be necessary.
- In step 2, $\mathrm{h}_{\text {esti }}$ is rounded-up to the nearest integer. This particular choice implies that the multiplications
by b2 and K can be replaced by "Additions" (or subtractions) and "Arithmetic Right Shift" operations. Therefore, a single multiplication per filter section (i.e. multiplication by " $W+\Delta W^{\prime \prime}$ ) is needed. In case of fixed-point arithmetic implementation, the optimum number of bits will be determined by the input signal dynamic range and the desired filter performances.


## DESIGN EXAMPLES

## EXAMPLE \#1

Design objectives: fs $=10 \mathrm{kHz} \mathrm{fo}=50 \mathrm{~Hz} \mathrm{fo} 2=2 \mathrm{kHz}$

$$
\mathrm{Att}_{\mathrm{dB}}=35 \mathrm{~dB} \quad \mathrm{Bw}_{\text {Goal }}=10 \quad \alpha=1
$$

$$
\rightarrow \mathrm{h}_{\mathrm{esti}} \approx 7.05 \quad \mathrm{~h}=7 \quad \mathrm{~b} 2 \approx 0.992 \quad \mathrm{~K} \approx 3.910^{-3}
$$

$$
\mathrm{L}_{\mathrm{esti}} \approx 5.82 \mathrm{~L}=5 \quad \Delta \mathrm{~W}_{\mathrm{fol}} \approx 6.5510^{-5} \mathrm{~W}_{\mathrm{fol} 1} \approx-1.991
$$

$$
\Delta \mathrm{W}_{\mathrm{fo} 2} \approx 2.6210^{-3} \mathrm{~W}_{\mathrm{fo} 2} \approx-0.616
$$

The amplitude response of this example is shown on Fig. 3 and 4. The difference between the two responses is due to the approximation used as well as the high $\mathrm{fo}_{2}$ to $\mathrm{fo}_{1}$ ratio!

## EXAMPLE \#2

The amplitude response of this example is shown on Fig. 5

## PERFORMANCE EXAMPLE

## Notch Filter "Tuning-Time" Response

In the example presented on Fig. $6(\mathrm{~h}=6, \mathrm{~L}=5)$, the signal is an On-Off modulated Shirp $(400 \mathrm{~Hz} \rightarrow 1500 \mathrm{~Hz})$ while the noise is a strong carrier whose frequency steps on fixed frequencies $(1500 \rightarrow 500 \mathrm{~Hz} \rightarrow 1000 \mathrm{~Hz}$ ). Both $W$ and $\Delta W$ are continually adapted so as to filter-out the noise (carrier). This test shows the efficiency of the tunable digital notch filter as well as its fast response time ( $\approx 4$ ms ).


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$$
\begin{aligned}
& \text { Design objectives: fs }=10 \mathrm{kHz} \quad \text { fo }=400 \mathrm{~Hz} \\
& \mathrm{Att}_{\mathrm{dB}}=40 \mathrm{~dB} \quad \mathrm{Bw}_{\text {Goal }}=40 \quad \alpha=0.625 \\
& \rightarrow \mathrm{~h}_{\text {esti }} \approx 6.30 \mathrm{~h}=6 \quad \mathrm{~b} 2 \approx 0.984 \quad \mathrm{~K} \approx 7.810^{-3} \\
& \mathrm{~L}_{\text {esti }} \approx 13.1 \mathrm{~L}=13 \quad \Delta \mathrm{~W} \approx 7.8510^{-4} \quad \mathrm{~W} \approx-1.922
\end{aligned}
$$



Fig. 1


Fig. 2 Cont. line: W, dotted line: W $+\Delta \mathrm{W}$, fo $=150 \mathrm{~Hz}$
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Fig. 3 Example \#1, fo $=50 \mathrm{~Hz}, \mathrm{~W}=-1.991$


Fig. 4 Example \#1, fo $=2000 \mathrm{~Hz}, \mathrm{~W}=-0.616$
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Fig. 5 Example \#2, fo $=400 \mathrm{~Hz}, \mathrm{~W}=-1.922$



Fig. 6 Top: Signal + Noise (SNR $\approx-17 \mathrm{~dB}$ ), Middle: W(-1.166, $-1.887,-1.605$ ), Bottom: Notch Filter Output Signal

